An Introduction to Perturbation Methods in Macroeconomics

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Introduction

- Numerous problems in macroeconomics involve functional equations of the form:

$$\mathcal{H}(d) = \mathbf{0}$$

- Examples: Value Function, Euler Equations.

- Regular equations are particular examples of functional equations.

- How do we solve functional equations?

Two Main Approaches

1. Projection Methods:

$$d^n(x,\theta) = \sum_{i=0}^{n} \theta_i \Psi_i(x)$$

   We pick a basis $\{\Psi_i(x)\}_{i=0}^{\infty}$ and "project" $\mathcal{H}(\cdot)$ against that basis.

2. Perturbation Methods:

$$d^n(x,\theta) = \sum_{i=0}^{n} \theta_i (x - x_0)^i$$

   We use implicit-function theorems to find coefficients θ_i.

Intuition of Perturbation Methods

- Most complicated problems have particular cases that are easy to solve.

- Often, we can use the solution to the particular case as a building block of the general solution.

- Very successful in physics.

- Judd and Guu (1993) showed how to apply it to economic problems.

A Simple Example

- Imagine we want to find the (possible more than one) roots of:

  $$x^3 - 4.1x + 0.2 = 0$$

  such that $x < 0$.

- This a tricky, cubic equation.

- How do we do it?

Main Idea

- Transform the problem rewriting it in terms of a small perturbation parameter.

- Solve the new problem for a particular choice of the perturbation parameter.

- Use the previous solution to approximate the solution of original the problem.

Step 1: Transform the Problem

- Write the problem into a perturbation problem indexed by a small parameter ε.

- This step is usually ambiguous since there are different ways to do so.

- A natural, and convenient, choice for our case is to rewrite the equation as:

$$x^3 - (4 + \varepsilon)\, x + 2\varepsilon = 0$$

where $\varepsilon \equiv 0.1$.

Step 2: Solve the New Problem

- Index the solutions as a function of the perturbation parameter $x = g(\varepsilon)$:

$$g(\varepsilon)^3 - (4 + \varepsilon) g(\varepsilon) + 2\varepsilon = 0$$

  and assume each of this solution is smooth (this can be shown to be the case for our particular example).

- Note that $\varepsilon = 0$ is easy to solve:

$$x^3 - 4x = 0$$

  that has roots $g(0) = -2, 0, 2$. Since we require $x < 0$, we take $g(0) = -2$.

Step 3: Build the Approximated Solution

- By Taylor's Theorem:

$$x = g(\varepsilon)|_{\varepsilon=0} = g(0) + \sum_{n=1}^{\infty} \frac{g^n(0)}{n!}\varepsilon^n$$

- Substitute the solution into the problem and recover the coefficients $g(0)$ and $\frac{g^n(0)}{n!}$for $n = 1, ...$ in an iterative way.

- Let's do it!

Zeroth -Order Approximation

- We just take $\varepsilon = 0$.
- Before we found that $g(0) = -2$.
- Is this a good approximation?

$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$
$$-8 + 8.2 + 0.2 = 0.4$$

- It depends!

First -Order Approximation

- Take the derivative of $g(\varepsilon)^3 - (4+\varepsilon)\, g(\varepsilon) + 2\varepsilon = 0$ with respect to ε:

$$3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4+\varepsilon)\, g'(\varepsilon) + 2 = 0$$

- Set $\varepsilon = 0$

$$3g(0)^2 g'(0) - g(0) - 4g'(0) + 2 = 0$$

- But we just found that $g(0) = -2$, so:

$$8g'(0) + 4 = 0$$

that implies $g'(0) = -\frac{1}{2}$.

First -Order Approximation

- By Taylor: $x = g\left(\varepsilon\right)\big|_{\varepsilon=0} \simeq g\left(0\right) + \frac{g^1(0)}{1!}\varepsilon^1$ or

$$x \simeq -2 - \frac{1}{2}\varepsilon$$

- For our case $\varepsilon \equiv 0.1$

$$x = -2 - \frac{1}{2} * 0.1 = -2.05$$

- Is this a good approximation?

$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$
$$-8.615125 + 8.405 + 0.2 = -0.010125$$

Second -Order Approximation

- Take the derivative of $3g(\varepsilon)^2 g'(\varepsilon) - g(\varepsilon) - (4+\varepsilon) g'(\varepsilon) + 2 = 0$ with respect to ε:

$$6g(\varepsilon)\left(g'(\varepsilon)\right)^2 + 3g(\varepsilon)^2 g''(\varepsilon) - g'(\varepsilon) - g'(\varepsilon) - (4+\varepsilon) g''(\varepsilon) = 0 \quad (1)$$

- Set $\varepsilon = 0$

$$6g(0)\left(g'(0)\right)^2 + 3g(0)^2 g''(0) - 2g'(0) - 4g''(0) = 0$$

- Since $g(0) = -2$ and $g'(0) = -\frac{1}{2}$, we get:

$$8g''(0) - 2 = 0$$

that implies $g''(0) = \frac{1}{4}$.

Second -Order Approximation

- By Taylor: $x = g(\varepsilon)|_{\varepsilon=0} \simeq g(0) + \frac{g^1(0)}{1!}\varepsilon^1 + \frac{g^2(0)}{2!}\varepsilon^2$ or

$$x \simeq -2 - \frac{1}{2}\varepsilon + \frac{1}{8}\varepsilon^2$$

- For our case $\varepsilon \equiv 0.1$

$$x = -2 - \frac{1}{2} * 0.1 + \frac{1}{8} * 0.01 = -2.04875$$

- Is this a good approximation?

$$x^3 - 4.1x + 0.2 = 0 \Rightarrow$$

$$-8.59937523242188 + 8.399875 + 0.2 = 4.997675781240329e - 004$$

Some Remarks

- The exact solution (up to machine precession of 14 decimal places) is $x = -2.04880884817015$.

- A second-order approximation delivers: $x = -2.04875$

- Relative error: 0.00002872393906.

- Yes, this was a rigged, but suggestive, example.

A Couple of Points to Remember

1. We transformed the original problem into a perturbation problem in such a way that the zeroth-order approximation has an analytical solution.

2. Solving for the first iteration involves a nonlinear (although trivial in our case) equation. All further iterations only require to solve a linear equation in one unknown.

An Application in Macroeconomics: Basic RBC

$$\max E_0 \sum_{t=0}^{\infty} \beta \{\log c_t\}$$

$$\begin{aligned} c_t + k_{t+1} &= e^{z_t} k_t^{\alpha} + (1-\delta) k_t, \; \forall \; t > 0 \\ z_t &= \rho z_{t-1} + \sigma \varepsilon_t, \;\; \varepsilon_t \sim \mathcal{N}(0,1) \end{aligned}$$

Equilibrium Conditions

$$\frac{1}{c_t} = \beta E_t \frac{1}{c_{t+1}} \left(1 + \alpha e^{z_{t+1}} k_{t+1}^{\alpha-1} - \delta\right)$$

$$c_t + k_{t+1} = e^{z_t} k_t^{\alpha} + (1-\delta) k_t$$

$$z_t = \rho z_{t-1} + \sigma \varepsilon_t$$

Computing the RBC

- The previous problem does not have a known "paper and pencil" solution.

- One particular case the model has a closed form solution: $\delta = 1$.

- Why? Because, the income and the substitution effect from a productivity shock cancel each other.

- Not very realistic but we are trying to learn here.

Solution

- By "Guess and Verify"

$$c_t = (1 - \alpha\beta)\, e^{z_t} k_t^{\alpha}$$
$$k_{t+1} = \alpha\beta e^{z_t} k_t^{\alpha}$$

- How can you check? Plug the solution in the equilibrium conditions.

Another Way to Solve the Problem

- Now let us suppose that you missed the lecture where "Guess and Verify" was explained.

- You need to compute the RBC.

- What you are searching for? A policy functions for consumption:

$$c_t = c\left(k_t, z_t\right)$$

and another one for capital:

$$k_{t+1} = k\left(k_t, z_t\right)$$

Equilibrium Conditions

- We substitute in the equilibrium conditions the budget constraint and the law of motion for technology.

- Then, we have the equilibrium conditions:

$$\frac{1}{c(k_t, z_t)} = \beta E_t \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t)^{\alpha-1}}{c(k(k_t, z_t), \rho z_t + \sigma \varepsilon_{t+1})}$$
$$c(k_t, z_t) + k(k_t, z_t) = e^{z_t} k_t^{\alpha}$$

- The Euler equation is the equivalent of $x^3 - 4.1x + 0.2 = 0$ in our simple example, and $c(k_t, z_t)$ and $k(k_t, z_t)$ are the equivalents of x.

A Perturbation Approach

- You want to transform the problem.

- Which perturbation parameter? standard deviation σ.

- Why σ?

- Set $\sigma = 0 \Rightarrow$deterministic model, $z_t = 0$ and $e^{z_t} = 1$.

Taylor's Theorem

- We search for policy function $c_t = c(k_t, z_t; \sigma)$ and $k_{t+1} = k(k_t, z_t; \sigma)$.

- Equilibrium conditions:

$$
\begin{aligned}
E_t\left(\frac{1}{c(k_t, z_t; \sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t, z_t; \sigma)^{\alpha - 1}}{c(k(k_t, z_t; \sigma), \rho z_t + \sigma \varepsilon_{t+1}; \sigma)}\right) &= 0 \\
c(k_t, z_t; \sigma) + k(k_t, z_t; \sigma) - e^{z_t} k_t^{\alpha} &= 0
\end{aligned}
$$

- We will take derivatives with respect to k_t, z_t, and σ.

Asymptotic Expansion $c_t = c(k_t, z_t; \sigma)|_{k,0,0}$

$$
\begin{aligned}
c_t \;=\;& c(k, 0; 0) \\
& + c_k(k, 0; 0)(k_t - k) + c_z(k, 0; 0) z_t + c_\sigma(k, 0; 0)\sigma \\
& + \frac{1}{2} c_{kk}(k, 0; 0)(k_t - k)^2 + \frac{1}{2} c_{kz}(k, 0; 0)(k_t - k) z_t \\
& + \frac{1}{2} c_{k\sigma}(k, 0; 0)(k_t - k)\sigma + \frac{1}{2} c_{zk}(k, 0; 0) z_t (k_t - k) \\
& + \frac{1}{2} c_{zz}(k, 0; 0) z_t^2 + \frac{1}{2} c_{z\sigma}(k, 0; 0) z_t \sigma \\
& + \frac{1}{2} c_{\sigma k}(k, 0; 0)\sigma(k_t - k) + \frac{1}{2} c_{\sigma z}(k, 0; 0)\sigma z_t \\
& + \frac{1}{2} c_{\sigma^2}(k, 0; 0)\sigma^2 + \ldots
\end{aligned}
$$

Asymptotic Expansion $k_{t+1} = k(k_t, z_t; \sigma)|_{k,0,0}$

$$
\begin{aligned}
k_{t+1} = \; & k(k, 0; 0) \\
& + k_k(k, 0; 0)\, k_t + k_z(k, 0; 0)\, z_t + k_\sigma(k, 0; 0)\, \sigma \\
& + \frac{1}{2} k_{kk}(k, 0; 0)(k_t - k)^2 + \frac{1}{2} k_{kz}(k, 0; 0)(k_t - k)\, z_t \\
& + \frac{1}{2} k_{k\sigma}(k, 0; 0)(k_t - k)\, \sigma + \frac{1}{2} k_{zk}(k, 0; 0)\, z_t (k_t - k) \\
& + \frac{1}{2} k_{zz}(k, 0; 0)\, z_t^2 + \frac{1}{2} k_{z\sigma}(k, 0; 0)\, z_t \sigma \\
& + \frac{1}{2} k_{\sigma k}(k, 0; 0)\, \sigma (k_t - k) + \frac{1}{2} k_{\sigma z}(k, 0; 0)\, \sigma z_t \\
& + \frac{1}{2} k_{\sigma^2}(k, 0; 0)\, \sigma^2 + \ldots
\end{aligned}
$$

Comment on Notation

- From now on, to save on notation, I will just write

$$F\left(k_t, z_t; \sigma\right) = E_t \left[\begin{array}{c} \frac{1}{c(k_t,z_t;\sigma)} - \beta \frac{\alpha e^{\rho z_t + \sigma \varepsilon_{t+1}} k(k_t,z_t;\sigma)^{\alpha-1}}{c(k(k_t,z_t;\sigma),\rho z_t + \sigma \varepsilon_{t+1};\sigma)} \\ c\left(k_t, z_t; \sigma\right) + k\left(k_t, z_t; \sigma\right) - e^{z_t} k_t^{\alpha} \end{array} \right] = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$$

- Note that:

$$F\left(k_t, z_t; \sigma\right) = H\left(c\left(k_t, z_t; \sigma\right), c\left(k\left(k_t, z_t; \sigma\right), z_{t+1}; \sigma\right), k\left(k_t, z_t; \sigma\right), k_t, z_t; \sigma\right)$$

- I will use H_i to represent the partial derivative of H with respect to the i component and drop the evaluation at the steady state of the functions when we do not need it.

Zeroth -Order Approximation

- First, we evaluate $\sigma = 0$:

$$F(k_t, 0; 0) = 0$$

- Steady state:

$$\frac{1}{c} = \beta \frac{\alpha k^{\alpha-1}}{c}$$

or,

$$1 = \alpha \beta k^{\alpha-1}$$

Steady State

- Then:

$$c = c(k, 0; 0) = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}$$
$$k = k(k, 0; 0) = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

- How good is this approximation?

First -Order Approximation

- We take derivatives of $F(k_t, z_t; \sigma)$ around $k, 0$, and 0.

- With respect to k_t:

$$F_k(k, 0; 0) = 0$$

- With respect to z_t:

$$F_z(k, 0; 0) = 0$$

- With respect to σ:

$$F_\sigma(k, 0; 0) = 0$$

Solving the System

Remember that:

$$F(k_t, z_t; \sigma) = H\left(c(k_t, z_t; \sigma), c(k(k_t, z_t; \sigma), z_{t+1}; \sigma), k(k_t, z_t; \sigma), k_t, z_t; \sigma\right)$$

Then:

$$F_k(k, 0; 0) = H_1 c_k + H_2 c_k k_k + H_3 k_k + H_4 = 0$$

$$F_z(k, 0; 0) = H_1 c_z + H_2 (c_k k_z + c_k \rho) + H_3 k_z + H_5 = 0$$

$$F_\sigma(k, 0; 0) = H_1 c_\sigma + H_2 (c_k k_\sigma + c_\sigma) + H_3 k_\sigma + H_6 = 0$$

Solving the System I

- Note that:

$$
\begin{aligned}
F_k(k,0;0) &= H_1 c_k + H_2 c_k k_k + H_3 k_k + H_4 = 0 \\
F_z(k,0;0) &= H_1 c_z + H_2 (c_k k_z + c_k \rho) + H_3 k_z + H_5 = 0
\end{aligned}
$$

  is a quadratic system of four equations on four unknowns: c_k, c_z, k_k, and k_z.

- Procedures to solve quadratic systems: Uhlig (1999).

- Why quadratic? Stable and unstable manifold.

Solving the System II

- Note that:

$$F_\sigma\left(k,0;0\right)=H_1c_\sigma+H_2\left(c_kk_\sigma+c_\sigma\right)+H_3k_\sigma+H_6=0$$

  is a linear, and homogeneous system in c_σ and k_σ.

- Hence

$$c_\sigma=k_\sigma=0$$

Comparison with Linearization

- After Kydland and Prescott (1982) a popular method to solve economic models has been the use of a LQ approximation.

- Close relative: linearization of equilibrium conditions.

- When properly implemented linearization, LQ, and first-order perturbation are equivalent.

- Advantages of linearization:

  1. Theorems.

  2. Higher order terms.

Second -Order Approximation

- We take second-order derivatives of $F(k_t, z_t; \sigma)$ around $k, 0$, and 0:

$$\begin{aligned}
F_{kk}(k, 0; 0) &= 0 \\
F_{kz}(k, 0; 0) &= 0 \\
F_{k\sigma}(k, 0; 0) &= 0 \\
F_{zz}(k, 0; 0) &= 0 \\
F_{z\sigma}(k, 0; 0) &= 0 \\
F_{\sigma\sigma}(k, 0; 0) &= 0
\end{aligned}$$

- Remember Young's theorem!

Solving the System

- We substitute the coefficients that we already know.
- A linear system of 12 equations on 12 unknowns. Why linear?
- Cross-terms $k\sigma$ and $z\sigma$ are zero.
- Conjecture on all the terms with odd powers of σ.

Correction for Risk

- We have a term in σ^2.

- Captures precautionary behavior.

- We do not have certainty equivalence any more!

- Important advantage of second order approximation.

Higher Order Terms

- We can continue the iteration for as long as we want.

- Often, a few iterations will be enough.

- The level of accuracy depends on the goal of the exercise: Fernández-Villaverde, Rubio-Ramírez, and Santos (2005).

A Computer

- In practice you do all this approximations with a computer.

- Burden: analytical derivatives.

- Why are numerical derivatives a bad idea?

- More theoretical point: do the derivatives exist? (Santos, 1992).

Code

- This afternoon there will be some demonstrations by Juilliard and Levin.

- First and second order: `Matlab` and `Dynare`.

- Higher order: `Mathematica`, `Fortran` code by Jinn and Judd.

An Example

- Let me run a second order approximation.

- Our choices

Calibrated Parameters

Parameter	β	α	ρ	σ
Value	0.99	0.33	0.95	0.01

Computation

- Steady State:

$$c = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.388069$$
$$k = (\alpha\beta)^{\frac{1}{1-\alpha}} = 0.1883$$

- First order components.

$$\begin{array}{ll} c_k(k,0;0) = 0.680101 & k_k(k,0;0) = 0.1883 \\ c_z(k,0;0) = 0.388069 & k_k(k,0;0) = 0.33 \\ c_\sigma(k,0;0) = 0 & k_k(k,0;0) = 0 \end{array}$$

Comparison

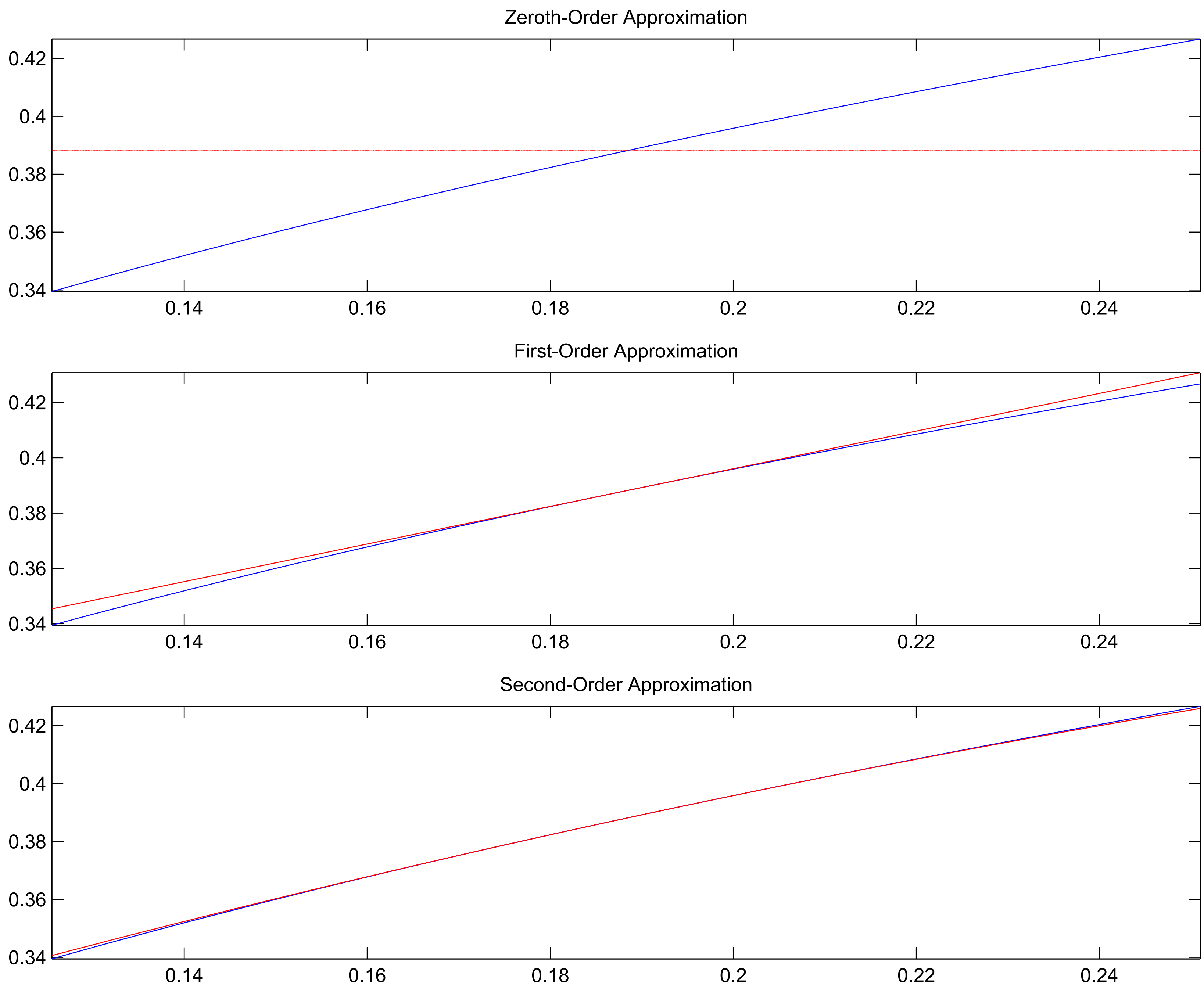
$$c_t = 0.6733 e^{z_t} k_t^{0.33}$$
$$c_t \simeq 0.388069 + 0.680101\,(k_t - k) + 0.388069 z_t$$

and:

$$k_{t+1} = 0.3267 e^{z_t} k_t^{0.33}$$
$$k_{t+1} \simeq 0.1883 + 0.1883\,(k_t - k) + 0.33 z_t$$

Second-Order Terms

$$
\begin{aligned}
c_{kk}(k,0;0) &= -2.41990 & k_{kk}(k,0;0) &= -1.1742 \\
c_{kz}(k,0;0) &= 0.680099 & k_{kz}(k,0;0) &= 0.330003 \\
c_{k\sigma}(k,0;0) &= 0. & k_{k\sigma}(k,0;0) &= 0 \\
c_{zz}(k,0;0) &= 0.388064 & k_{zz}(k,0;0) &= 0.188304 \\
c_{z\sigma}(k,0;0) &= 0 & k_{z\sigma}(k,0;0) &= 0 \\
c_{\sigma^2}(k,0;0) &= 0 & k_{\sigma^2}(k,0;0) &= 0
\end{aligned}
$$



Zeroth-Order Approximation
First-Order Approximation
Second-Order Approximation
0.34
0.36
0.38
0.4
0.42
0.14
0.16
0.18
0.2
0.22
0.24

Non Local Accuracy test (Judd, 1992, and Judd and Guu, 1997)

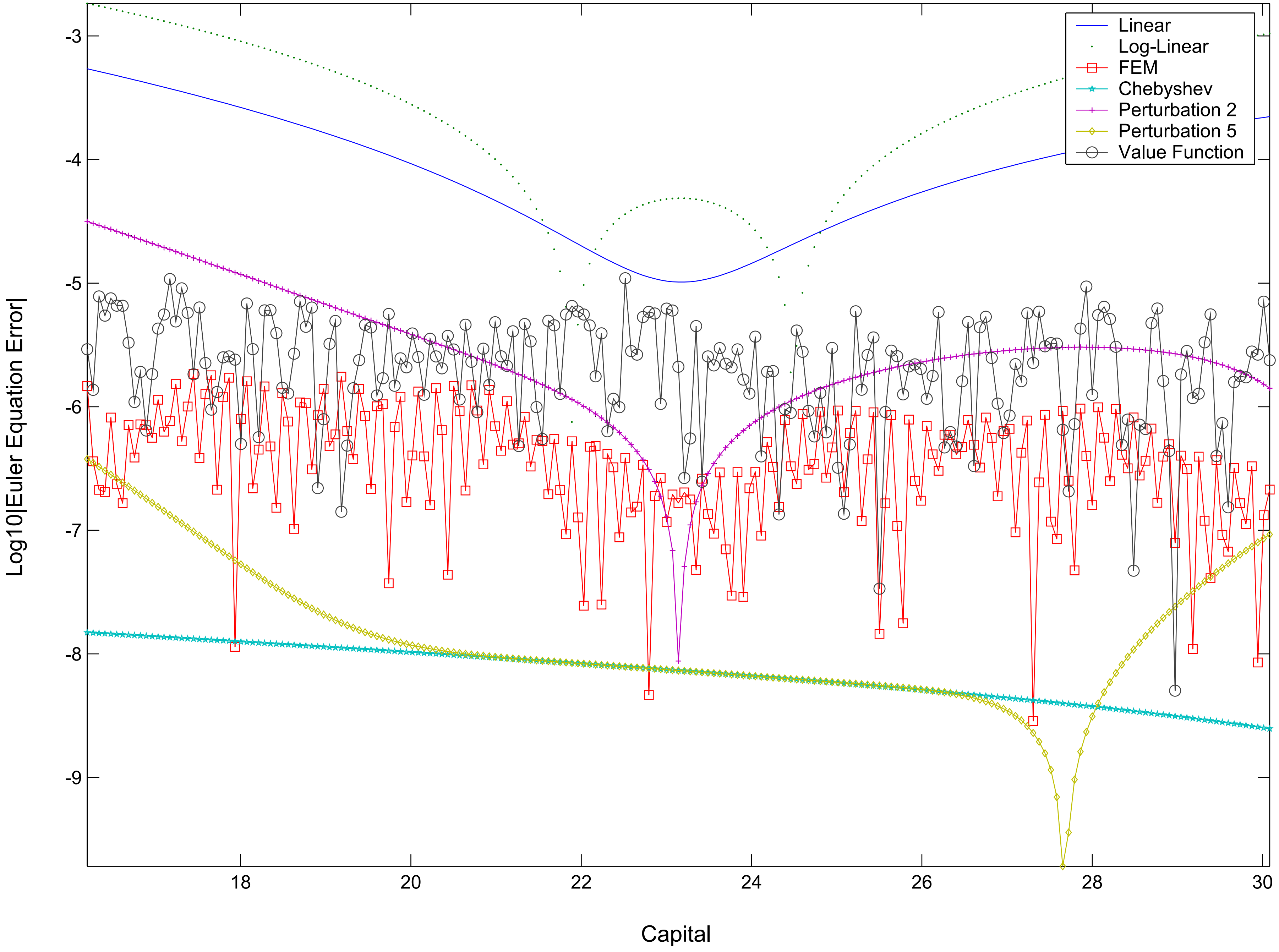
Given the Euler equation:

$$\frac{1}{c^i(k_t, z_t)} = E_t\left(\frac{\alpha e^{z_{t+1}} k^i(k_t, z_t)^{\alpha-1}}{c^i\left(k^i(k_t, z_t), z_{t+1}\right)}\right)$$

we can define:

$$EE^i(k_t, z_t) \equiv 1 - c^i(k_t, z_t)\, E_t\left(\frac{\alpha e^{z_{t+1}} k^i(k_t, z_t)^{\alpha-1}}{c^i\left(k^i(k_t, z_t), z_{t+1}\right)}\right)$$

Figure 5.4.1 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007



Changes of Variables

- We approximated our solution in levels.
- You may have read about loglinearization.
- Note that:

$$x_t = xe^{\log \frac{x_t}{x}} = xe^{\widehat{x}_t}$$

- We can substitute all variables x_t by $\widehat{x}_t$.
- Why can loglinearization be a good idea?

Judd (2002)

- Why stop there? Why not in powers of the state variables?

- Judd (2002) has provided methods for changes of variables.

- We apply and extend ideas to the stochastic neoclassical growth model.

A General Transformation

- We look at solutions of the form:

$$\begin{aligned} c^{\mu} - c_0^{\mu} &= a\left(k^{\zeta} - k_0^{\zeta}\right) + cz^{\varphi} \\ k'^{\gamma} - k_0^{\gamma} &= c\left(k^{\zeta} - k_0^{\zeta}\right) + dz^{\varphi} \end{aligned}$$

  where $\varphi \geq 1$.

- Note that:

  1. If γ, ζ, μ and φ are 1 we get the linear representation.

  2. As γ, ζ and μ tend to zero and φ is equal to 1 we get the loglinear approximation.

Theory

- The first order solution can be written as

$$f(x) \simeq f(a) + (x-a) f'(a)$$

- Expand $g(y) = h(f(X(y)))$ around $b = Y(a)$, where $X(y)$ is the inverse of $Y(x)$.

- Then:

$$g(y) = h(f(X(y))) = g(b) + g_\alpha(b)(Y^\alpha(x) - b^\alpha)$$

where $g_\alpha = h_A f_i^A X_\alpha^i$ comes from the application of the chain rule.

- From this expression it is easy to see that if we have computed the values of f_i^A, then it is straightforward to find the value of g_α.

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Figure 6.2.1 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007

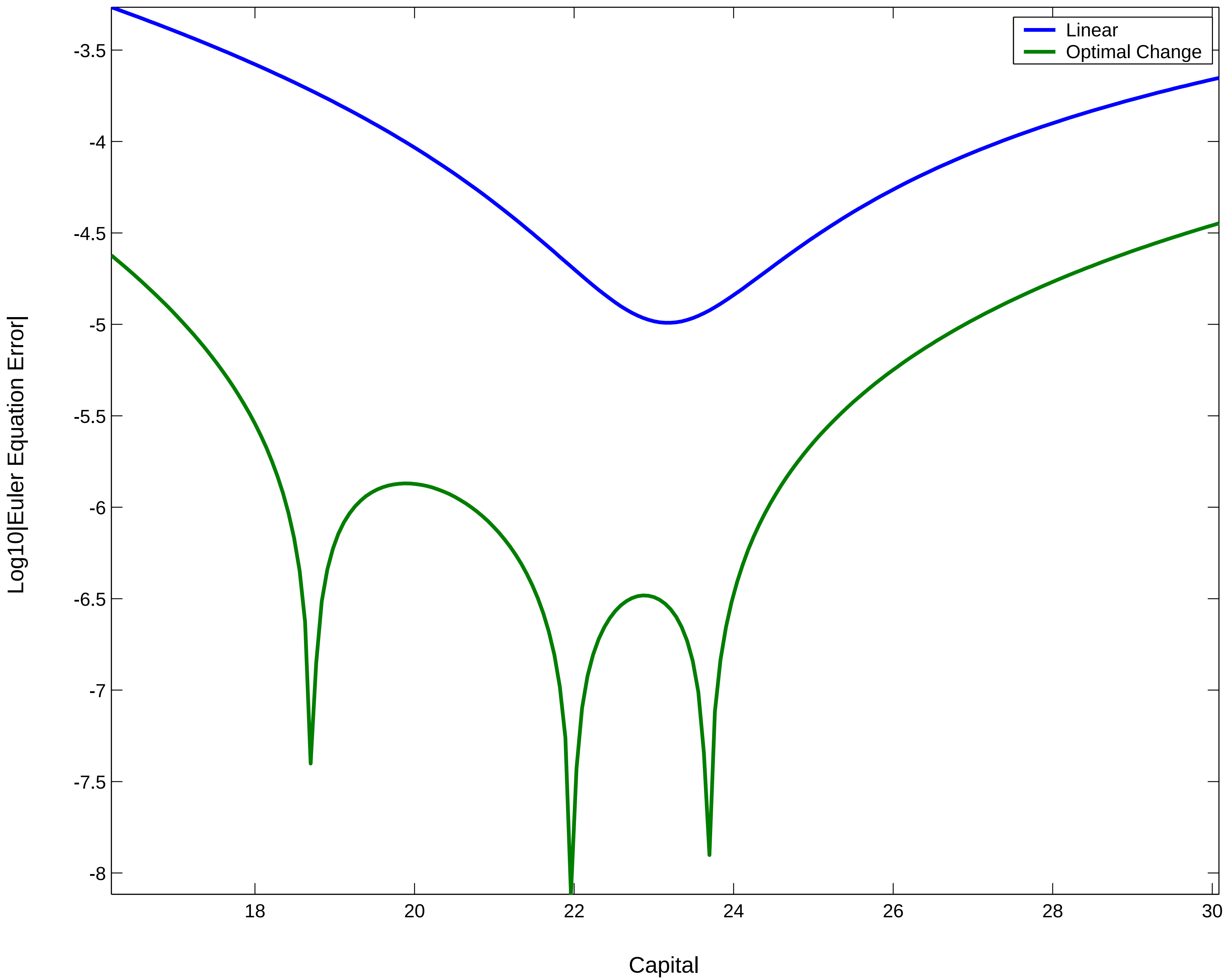
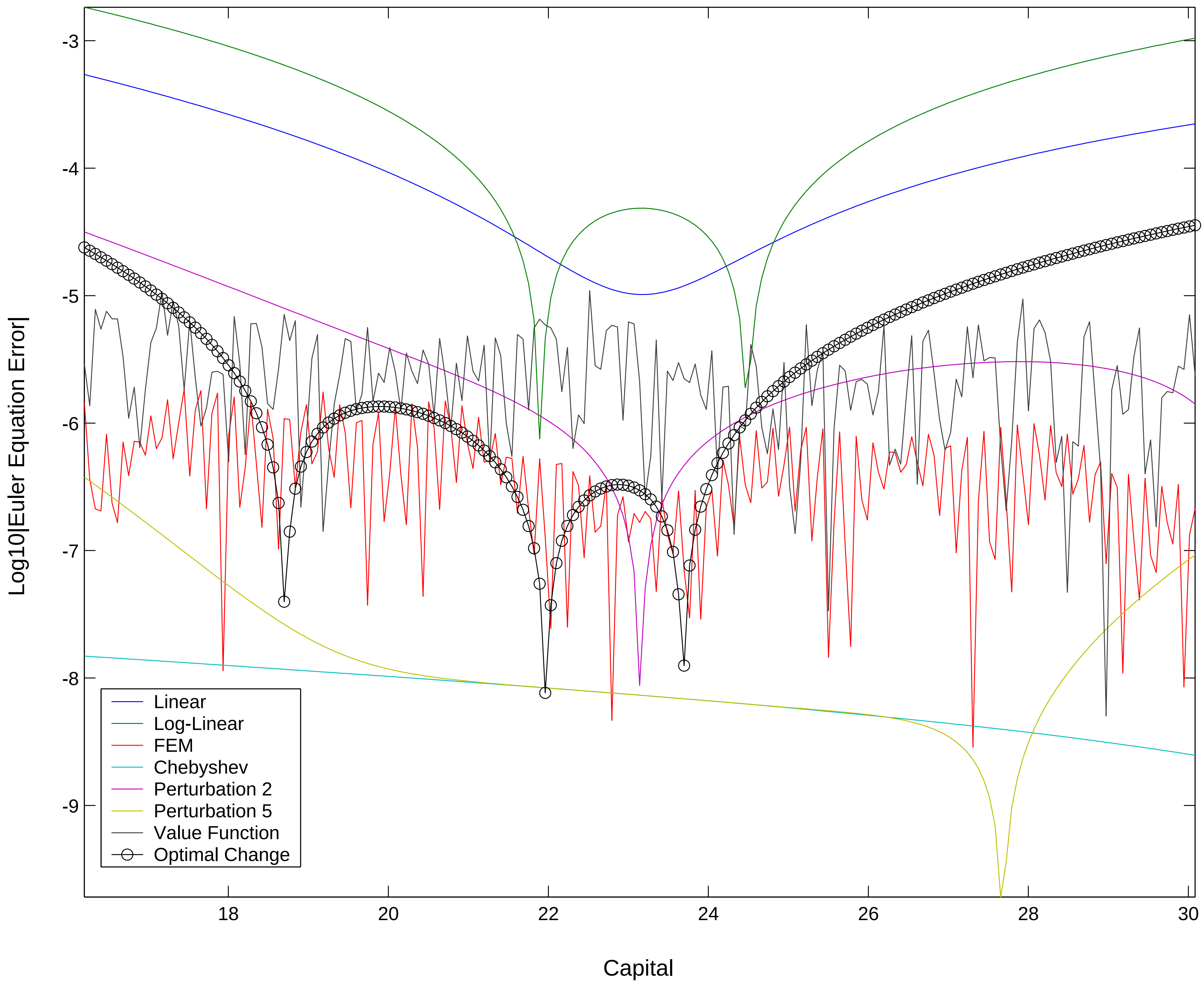


Figure 6.2.2 : Euler Equation Errors at z = 0, τ = 2 / σ = 0.007



En Savoir Plus I

- General Perturbation theory: *Advanced Mathematical Methods for Scientists and Engineers: Asymptotic Methods and Perturbation Theory* by Carl M. Bender, Steven A. Orszag.

- Perturbation in Economics:

  1. "Perturbation Methods for General Dynamic Stochastic Models" by Hehui Jin and Kenneth Judd.

  2. "Perturbation Methods with Nonlinear Changes of Variables" by Kenneth Judd.

En Savoir Plus II

- A gentle introduction: “Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function” by Martín Uribe and Stephanie Schmitt-Grohe.

- Solving quadratic equations: A Toolkit for Analysing Nonlinear Dynamic Stochastic Models Easily” by Harald Uhlig.