Using projection methods to compute equilibrium in stochastic models with overlapping generations

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Tutorial on models with heterogeneous agents

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An abstract environment, FREE

- State space $\mathcal{Y} = \Theta \times \mathcal{S}$,
- policy and pricing functions $F(\theta, s)$ satisfy

$$g\left(\theta, s, F(\theta, s), \mathbf{E}_s\left[h\left(F(\theta, s), F\left(F(\theta, s), \tilde{s}\right), \tilde{s}\right)\right]\right) = 0$$

  for all $(\theta, s) \in \Theta \times \mathcal{S}$.
- g and h are (known) smooth functions
- Want algorithms to approximate F

Why projection methods ?

- Easy to formulate, independenty of number of assets, utility and production functions (as long as first order conditions characterize optimality
- Clear economic interpretation: approximate infinite economies by economies with large finite horizon
- Programming costs fairly high, but once toolbox is built up, very mechanical
- Comparative advantage compared to perturbations methods when solution is 'very non-linear'

A solution method - time iteration collocation

0: Select a d-box Θ, a family of functions $\widehat{F}$ which can be parametrized by numbers

1: Select a finite grid $\mathcal{G} \subset \Theta$ of collocation points and the parameters $\xi(0)$ for a starting $\widehat{F}^0$

2: Given parameters $\xi(n)$ and thus the function $\widehat{F}^n$, $\forall \theta \in \mathcal{G}$, $\forall s \in \mathcal{S}$, solve system

$$g\left(\theta, s, x, \mathbf{E}_s\left[h\left(x, \widehat{F}^n(x, \tilde{s}), \tilde{s}\right)\right]\right) = 0$$

for the unknown x

3: Compute the new coefficients $\xi(n+1)$ by interpolation of the solutions in 2

4: Check some stopping criterion, if not satisfied, go to 2

5: Conduct error analysis

What is needed

- **Approximation of policy functions**
  - **How to select** Θ
  - **How to select** $\mathcal{G}$
  - **Interpolation, picking** $\xi(0)$
- **Solving nonlinear equations**

Rest of talk

* A SOLG model
* High-dimensional interpolation
* Numerical examples

Model

- **Exogenous shocks, z_t, follow a Markov process with finite support $\mathcal{Z}$ – histories of shocks are $z^t = (z_1, \dots, z_t)$**
- **An individual, z^t, lives for N periods, has labor endowment l^n at age n and utility over consumption**
- **Recursive utility (Kreps & Porteus (1978) and Epstein & Zin (1989))**

$$U^s(c, z^t) = \left\{ \left[c^s(z^t)\right]^\rho + \beta \left[\sum_{z_{t+1}} \pi(z_{t+1}|z_t) \left[U^s(c, z^{t+1})\right]^\sigma \right]^{\frac{\rho}{\sigma}} \right\}^{\frac{1}{\rho}}$$

with $U^s(c, z^{s+N-1}) = c^s(z^{s+N-1})$

Model (cont.)

- Single firm has production function

$$f(K,L) = \eta(z_t)K^{\alpha}L^{1-\alpha}, \text{ stochastic depreciation } \delta(z_t)$$

- Household s sells labor to firm and rents out capital, buys bonds, his budget constraint is

$$c_t^s + a_t^s + q_t b_t^s = r_t a_{t-1}^s + l^{t-s+1} w_t + b_{t-1}^s$$

- Competitive equilibrium consists of prices and choices such that households maximize utility, firm maximizes profits and markets clear

Functional Rational Expectations Equilibrium

(Smooth) functions $a_z^i, b_z^i : \Theta_z \to \mathbb{R}$ as well as $q_z : \Theta_z \to \mathbb{R}_+$ and $U_z^i : \Theta_z \to \mathbb{R}$ are defined on $d = N-1$ dimensional boxes Θ_z, for all $z \in \mathcal{Z}$, such that for all $\theta \in \Theta$, $z \in \mathcal{Z}$, $i = 1, ..., N-1$,

$$\Gamma_z^i(\theta) E_z \left[\left(\frac{c_{\tilde{z}}^{i+1}(\theta'_{\tilde{z}})}{c_z^i(\theta)} \right)^{\rho-1} R_{\tilde{z}}(\theta'_{\tilde{z}}) \left(U_{\tilde{z}}^{i+1}(\theta'_{\tilde{z}}) \right)^{\sigma-\rho} \right] = 1$$

$$\Gamma_z^i(\theta) E_z \left[\left(\frac{c_{\tilde{z}}^{i+1}(\theta'_{\tilde{z}})}{c_z^i(\theta)} \right)^{\rho-1} \frac{1}{q_z(\theta)} \left(U_{\tilde{z}}^{i+1}(\theta'_{\tilde{z}}) \right)^{\sigma-\rho} \right] = 1,$$

where

$$\Gamma_z^i(\theta) = \beta \left[E_z \left(U_{\tilde{z}}^{i+1}(\theta'_{\tilde{z}}) \right)^{\sigma} \right]^{\frac{\rho}{\sigma}-1}$$

$$c^i(\theta, z) = \theta_i + l^i(z) w(\theta, z) - a_z^i(\theta) - q_z(\theta) b_z^i(\theta) \quad N > i > 1$$

$$R_z(\theta) = 1 + f_K(\theta_1, \sum_{i=1}^{N} l^i(z), z), \; w(\theta, z) = f_L(\theta_1, \sum_{i=1}^{N} l^i(z), z)$$

and

$$\theta'_{\tilde{z}} = (\sum_{n=1}^{N-1} a_z^n(\theta), a_z^1(\theta)R_{\tilde{z}} + b_z^1(\theta), \ldots, a_z^{N-2}(\theta)R_{\tilde{z}} + b_z^{N-2}(\theta))$$

Some refinements of basic time-iteration algorithm

- Use different box for each shock Θ_z
- For range of aggregate capital use deterministic steady states plus minus x percent.
- Adjust size of boxes along time-iteration

Approximating the policy functions

- We want to approximate $f : [-1, 1]^d \to \mathbb{R}$, using sparse grids
- Number of points in sparse grids grow slowly with dimension, approximation error of full grids is preserved up to logarithmic constant
- Use Smolyak's method for polynomial interpolation; see Barthelmann, Novak and Ritter (Advances of Computational Mathematics 12, 1999) (or Krueger and Kubler (2004) for the olg model)
- Fortran 90 code available from me; matlab code available at http://matlabdb.mathematik.uni-stuttgart.de

Sparse grids – basic idea

- **Define nested sequence $\chi^i \subset [-1,1]$, i=1,2,...**
- **With $\chi^0 = \emptyset$, $\chi^i = \chi^{i-1} \cup \chi^i_\Delta$ and $\mathcal{H}(d-1,d) = \emptyset$, define recursively**

$$\mathcal{H}(q,d) = \mathcal{H}(q-1,d) \cup \Delta\mathcal{H}(q,d) \tag{1}$$

with

$$\Delta\mathcal{H}(q,d) = \bigcup_{(i_1,\ldots,i_d)\in\mathbb{Z}^d_{++}:\sum_l i_l = q} (\chi^{i_1}_\Delta \times \ldots \times \chi^{i_d}_\Delta)$$

Sparse grids – Simplest example

- **Suppose** $\chi^1 = \{0\}$, $\chi^2_\Delta = \{-1, 1\}$, $\chi^3_\Delta = \{-0.5, 0.5\}$, $\chi^4_\Delta = \{-0.2, 0.2\}$
- $\mathcal{H}(2,2) = \{(0,0)\}$, $\mathcal{H}(3,2) = \{(0,0)\} \cup \chi^2_\Delta \times \chi^1_\Delta \cup \chi^1_\Delta \times \chi^2_\Delta$
- **More interestingly**

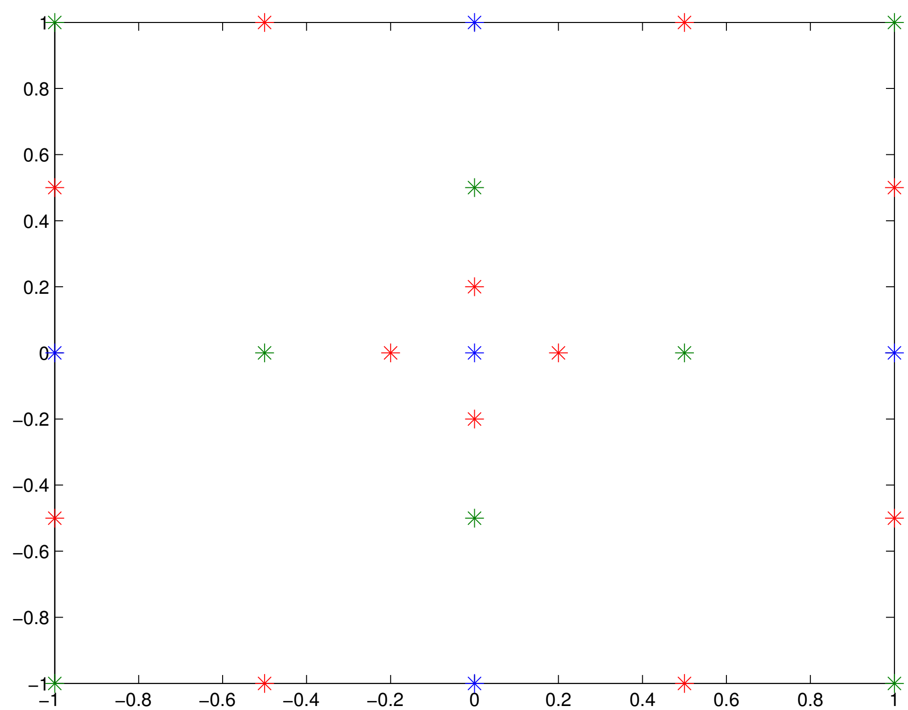
$$\mathcal{H}(4,2) = \mathcal{H}(3,2) \cup \chi^3_\Delta \times \chi^1_\Delta \cup \chi^1_\Delta \times \chi^3_\Delta \cup \chi^2_\Delta \times \chi^2_\Delta$$

and

$$\mathcal{H}(5,2) = \mathcal{H}(4,2) \cup \chi^3_\Delta \times \chi^2_\Delta \cup \chi^2_\Delta \times \chi^3_\Delta \cup \chi^4_\Delta \times \chi^1_\Delta \cup \chi^1_\Delta \times \chi^4_\Delta$$

- **Number of points in** $\mathcal{H}(d+n, d)$ **grows slowly in** d

Sparse grids – Illustration



Smolyak's grid

- **Define** $m_i = 2^{i-1} + 1$ **to be the total number of elements of set** χ^i
- **Choose** $\chi^1 = \{0\}$ **and for** $i > 1$, $\chi^i = \{x_1^i, ..., x_{m_i}^i\} \subset [-1, 1]$ **as the set of the extrema of the Chebychev polynomials**

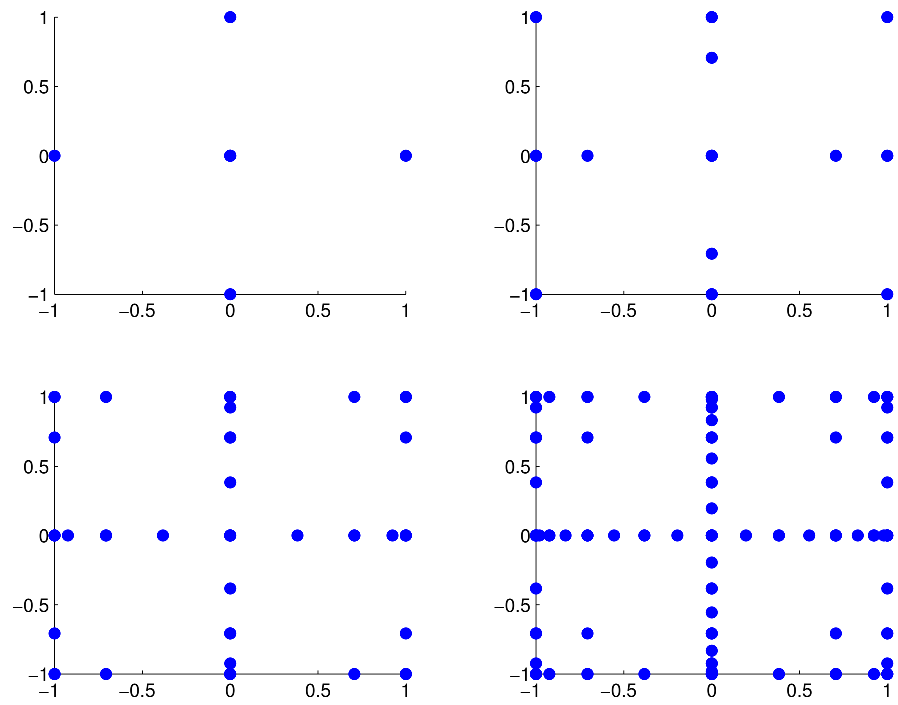
$$x_j^i = -cos\frac{\pi(j-1)}{m_i - 1} \quad j = 1, ..., m_i$$

- **So** $\chi^1 = \{0\}$, $\chi^2_\Delta = \{-1, 1\}$, $\chi^3_\Delta = \{\cos(\frac{3\pi}{4}), -\cos(\frac{3\pi}{4}\}$ **and**

$$\chi^4_\Delta = \{-\cos(\frac{\pi}{8}), -\cos(\frac{3\pi}{8}), \cos(\frac{\pi}{8}), \cos(\frac{3\pi}{8})\}$$

- **Number of points grows exponentially in** i, **but we will not go beyond** $i = 4$.

Smolyak's grids



Interpolation over $\mathcal{H}$, Intuition

- Denote by p^{ij} the tensor-product that interpolates f over $\chi^i \times \chi^j$. Consider $q = 4$, $d = 2$.
- There exist coefficients $\alpha_{ij} \in \mathbb{Z}$ to ensure that

  $$\mathcal{A}(4,2) = \alpha_{11}p^{11} + \alpha_{21}p^{21} + \alpha_{12}p^{12} + \alpha_{31}p^{31} + \alpha_{13}p^{13} + \alpha_{22}p^{22}$$

  interpolates f over

  $$\mathcal{H}(4,2) = \chi^1 \times \chi^1 \cup \chi^2 \times \chi^1 \cup \chi^1 \times \chi^2 \cup \chi^3 \times \chi^1 \cup \chi^1 \times \chi^3 \cup \chi^2 \times \chi^2$$

- This follows from the construction of χ^i − can easily compute the coefficients

Interpolation over $\mathcal{H}$, Equations

- **For general q, d, define $\mathbf{i} = (i_1, ..., i_d) \in \mathbb{Z}^d_{++}$ and let $|\mathbf{i}| = \sum_j i_j$. Rewrite the grid as follows**

$$\mathcal{H}(q,d) = \bigcup_{q-d+1 \leq |\mathbf{i}| \leq q} (\chi^{i_1} \times ... \times \chi^{i_d}),$$

- **For $q > d$, the Smolyak construction of an interpolating polynomial is then**

$$\mathcal{A}(q,d) = \sum_{q-d+1 \leq |\mathbf{i}| \leq q} (-1)^{q-|\mathbf{i}|} \begin{pmatrix} d-1 \\ q-|\mathbf{i}| \end{pmatrix} \alpha_{i_1,\ldots,i_d} (\mathcal{U}^{i_1} \otimes \ldots \otimes \mathcal{U}^{i_d})$$

Remarks

- $\mathcal{A}(d+2,d)$ **reproduces the polynomials** x_j^4, x_j^3, x_j^2, x_j, 1, $x_j^2 x_k^2$, $x_j^2 x_k$, $x_j x_k$.
  $\mathcal{A}(d+k,d)$ **is exact for polynomials up to degree** k
- **Given** $\mathcal{H}$, **Smolyak's construction gives least interpolating polynomial in the sense of deBoor and Ron.**
- **The number of points in** $\mathcal{H}(q,d)$ **is given by**
  $q = d+1 : 1+2d$
  $q = d+2 : 1+4d+4\frac{d(d-1)}{2}$
  $q = d+3 : 1+8d+16\frac{d(d-1)}{2}+8\frac{d(d-1)(d-2)}{6}$
- **See Barthelman et al (1999) for an error analysis**

Large N

- **In many applications one wants to interpret a period as a year. Most Americans live for more than 50 years after entering the labor market**
  $\Rightarrow N > 50$.
- **In the literature one can find two 'solutions'**
  - **Log-linearize (perturbation methods ?)**
  - **Very coarse approximation of transition function (Krusell and Smith (1996), Storesletten et al (2002), Gourinchas (2000))**

Time iteration collocation for large N

- For $d = 1$ use Chebychev polynomials
- Start with grid $\mathcal{H}(d+1, d)$ and interpolating functions $\mathcal{A}(d+1, d)$ to obtain a coarse approximation
- Refine by moving to grid $\mathcal{H}(d+2, d)$, $\mathcal{A}(d+2, d)$ using coarse results as a starting point
- Iterate for $q = d+2$ until desired error is achieved
- One iteration for $q = d+3$ on grid $\mathcal{H}(d+3, d)$

Evaluating the algorithms I: 'Standard examples

- **Consider cases** $N = 20$, $N = 30$, $N = 40$
- **Risk aversion and IES is** $\sigma = 1.5$, $\rho = 0.5$. **Patience is given by** $\beta = 0.95^{60/N}$. $l^n = 1$ **for** $n \leq 2/3N$ **and** $l^n = 0.25$ **otherwise. The production function is Cobb Douglas with** $\alpha = 0.3$. **We consider 2 cases for shocks:**
  - $\delta = 0.7$, $\eta \in \{0.85, 1.15\}$
  - $\delta \in \{0.7 + 0.15, 0.7 - 0.15\}$

Error criterion:

1 Report average and maximal error along one simulation of 50000 periods, using residual from Euler equations

2 Construct ϵ equilibrium and determine maximal necessary (linear quadratic) perturbations in preferences (Kubler and Schmedders (2005))

Results

Table 1: Sim. Errors and Running Times (2.2 GHz)		
	Case 1	Case 2
N=20, Max. Err.	6.9 (-4)	9.0 (-4)
N=20, Avg. Err.	1.0 (-4)	2.3 (-4)
running time	1 min, 16 sec	1 min, 32 sec
N=30, Max. Err.	4.1 (-3)	7.2 (-3)
N=30, Avg. Err.	0.9 (-3)	2.5 (-3)
running time	27 min, 12 sec	28 min, 2
N=40, Max. Err.	8.3 (-3)	1.2 (-2)
N=40, Avg. Err.	4.7 (-3)	4.1 (-3)
running time	2h 27 min	3h 51

Large equity premium and high curvature

- **Suppose $N = 9$, specify shocks and preferences to obtain sizable equity premium.**
- **Big shocks: $\delta \in \{0, 1\}$, $\eta \in \{0.8, 1.2\}$**
- **With $IES = 0.5$, calibrate risk aversion to match equity premium $\to \sigma = 25$.**
- **Maximal error in Euler equation for $q = d + 2$ is over 10^{-2} but for $q = d + 3$ decreases to below 10^{-3}**
- **Policy and pricing function exhibit strong non-linearities**