Solution of Leader-Follower Games

Sven Leyffer leyffer@mcs.anl.gov
Mathematics & Computer Science Division,
Argonne National Laboratory

Introduction: Nash & Stackelberg Games
Formulation of MPECs as NLPs
Convergence of NLP Approach
Multi-Leader-Follower Games
Conclusions
NASH & STACKELBERG GAMES
Recap: Nash Games

Nash Game: non-cooperative equilibrium of several players

\[ z_i^* \in \left\{ \begin{array}{l} \arg \min_{z_i} \quad b_i(\hat{z}) \\ \text{subject to} \quad c_i(z_i) \geq 0 \end{array} \right\} \quad \text{player } i \]

- \( \hat{z} = (z_1^*, \ldots, z_{i-1}^*, z_i, z_{i+1}^*, \ldots, z_l^*) \)
- All players are equal
Recap: Nash Games

Introduce slacks $s$, and form optimality conditions ...

\[ \nabla b(z) - \nabla c(z) \lambda = 0 \]
\[ s - c(z) = 0 \]
\[ 0 \leq \lambda \perp s \geq 0 \]

where

- $b(z) = (b_1(z), \ldots, b_k(z))$ & $c(z) = (c_1(z), \ldots, c_k(z))$
- $\perp$ means $\lambda^T s = 0$, either $\lambda_i > 0$ or $s_i > 0$

$y = (z, \lambda, s)$ \quad ... becomes ... \quad \left\{ \begin{array}{l}
    h(y) = 0 \\
    0 \leq y_1 \perp y_2 \geq 0
\end{array} \right.$

- Nonlinear complementarity problem (NCP)
- Robust large scale solvers exist: PATH
Stackelberg Games

Single dominant player (leader) & Nash followers

\[
\begin{align*}
\text{minimize} & \quad f(x, y) \\
\text{subject to} & \quad h(x, y) = 0 \\
& \quad 0 \leq y_1 \perp y_2 \geq 0
\end{align*}
\]

Nash game \((h(x, y) = 0)\) parameterized in leader’s variables \(x\)

Mathematical Program with Equilibrium Constraints (MPEC)
Mathematical Program with Equilibrium Constraints

Mathematical Program with Equilibrium Constraints (MPEC)

\[
\begin{align*}
\text{minimize} & \quad f(x) & \text{objective} \\
\text{subject to} & \quad c(x) \geq 0 & \text{constraints} \\
& \quad 0 \leq x_1 \perp x_2 \geq 0 & \text{complementarity}
\end{align*}
\]

where \( x = (x_0, x_1, x_2) \) ... partition of variables and

\[
0 \leq x_1 \perp x_2 \geq 0 \iff \text{either } x_{1i} = 0 \text{ or } x_{2i} = 0
\]

... equality constraints \( h(x) = 0 \) are no problem!
EXAMPLE: COMMODITIES FLOW IN NETWORK
Commodities Flow in a Network

- variables are flow on arcs & total flow:
  
  var x \{\text{ARCS, COMMODITIES}\} >= 0;
  var f \{(i,j) \text{ in ARCS}\}
  = \text{sum} \{k \text{ in COMMODITIES}\} x[i,j,k];

- each player (commodity) minimizes time in network
  
  minimize time \{k \text{ in COMMODITIES}\}:
  \text{sum} \{(i,j) \text{ in ARCS}\} (\alpha[i,j] \times f[i,j] + \beta[i,j] \times (f[i,j]/\kappa[i,j]/5)^5);

- subject to conservation of flow (supply/demand)
  
  subject to conserve \{i \text{ in NODES, k in COMMODITIES}\}:
  \text{sum} \{(i,j) \text{ in ARCS}\} x[i,j,k] <= \text{sum}\{(j,i) \text{ in ARCS}\} x[j,i,k] + b[i,k];
Flow of Commodities Through a Network

set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES;

param b {NODES, COMMODITIES} default 0;
param alpha {ARCS} >= 0;
param beta {ARCS} >= 0;
param kappa {ARCS} >= 0;

var x {ARCS, COMMODITIES} >= 0;
var f {(i,j) in ARCS}
   = sum {k in COMMODITIES} x[i,j,k];
Flow of Commodities Through a Network

minimize time \{k \in \text{COMMODITIES}\}:
\sum \{(i,j) \in \text{ARCS}\} (\alpha[i,j]*f[i,j] \\
+ \beta[i,j]*(f[i,j]/\kappa[i,j]/5)^5); \\

subject to \\
\text{conserve} \{i \in \text{NODES}, k \in \text{COMMODITIES}\}: \\
\sum \{(i,j) \in \text{ARCS}\} x[i,j,k] \\
\leq \sum\{(j,i) \in \text{ARCS}\} x[j,i,k] + b[i,k]; \\

\text{problem player} \{k \in \text{COMMODITIES}\}: \\
\text{time}[k], \quad \# \text{objective} \\
\{i \in \text{NODES}\} \text{conserve}[i,k], \quad \# \text{constraints} \\
\{(i,j) \in \text{ARCS}\} x[i,j,k], f; \quad \# \text{variables}
Nash Game: Complementarity Model

subject to

d_obj {(i,j) in ARCS, k in COMMODITIES}:

\[ 0 \leq x_{i,j,k} \text{ complements } \alpha_{i,j} + \beta_{i,j} \frac{f_{i,j}}{\kappa_{i,j}}^4 + p_{i,k} \geq p_{j,k}; \]

conserve {i in NODES, k in COMMODITIES}:

\[ 0 \leq p_{i,k} \text{ complements } \sum{(j,i) in ARCS} x_{j,i,k} + d_{i,k} \geq \sum{(i,j) in ARCS} x_{i,j,k}; \]

flow {(i,j) in ARCS} :

\[ f_{i,j} = \sum{k in COMMODITIES} x_{i,j,k} \text{ complements } f_{i,j}; \text{ ### not needed here} \]
Stackelberg Game: MPEC

- introduce toll arcs (faster, but carry cost)
- leader chooses tolls to maximize profit ...
  ... or minimize congestion
- followers move optimally (Wardrop’s principle)
Stackelberg Game: MPEC

set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES;

param b {NODES, COMMODITIES} default 0;  # Supply/demand
param alpha {ARCS} >= 0;  # Linear part
param beta {ARCS} >= 0;  # Nonlinear part
param kappa {ARCS} >= 0;  # Nonlinear part

var t {(i,j) in TOLL}  # Tolls
   >= tl[i,j], <= tu[i,j],
   := (tl[i,j] + tu[i,j])/2 ;

var x {ARCS, COMMODITIES};  # Flow on arcs
var p {NODES, COMMODITIES};  # Multipliers
var f {(i,j) in ARCS};  # Total flow
**Stackelberg Game: MPEC**

\[
\text{maximize revenue: } \sum_{(i,j) \in \text{TOLL}} t[i,j] * f[i,j];
\]

subject to

\[
\text{d_obj } \{(i,j) \in \text{ARCS}, k \in \text{COMMODITIES} \}:
\]
\[
0 \leq x[i,j,k] \text{ complements } \alpha[i,j] + \beta[i,j] \cdot (f[i,j]/\kappa[i,j])^4
\]
\[
+ (\text{if } (i,j) \in \text{TOLL then } t[i,j] \text{ else } 0)
\]
\[
+ p[i,k] \geq p[j,k];
\]

\[
\text{conserve } \{i \in \text{NODES}, k \in \text{COMMODITIES} \}:
\]
\[
0 \leq p[i,k] \text{ complements } \sum\{(j,i) \in \text{ARCS}\} x[j,i,k] + b[i,k]
\]
\[
\geq \sum\{(i,j) \in \text{ARCS}\} x[i,j,k];
\]

\[
\text{flow } \{(i,j) \in \text{ARCS}\}:
\]
\[
f[i,j] = \sum \{k \in \text{COMMODITIES}\} x[i,j,k];
\]
Solving Large Stackelberg Games

Toll pricing models get large ...

<table>
<thead>
<tr>
<th># nodes</th>
<th># variables</th>
<th># constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>112</td>
<td>109</td>
</tr>
<tr>
<td>25</td>
<td>5000</td>
<td>2500</td>
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... depending on arches etc ...
Solving Large Stackelberg Games

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... depending on arches etc ...

Exploit NCP solvers to set-up MPECs:

1. fix leader’s variables & solve NCP (Nash)
2. unfix leader’s variables & solve MPEC (Stackelberg)

⇒ orders of magnitude faster than MPEC ...
From Nash to Stackelberg with AMPL

```plaintext
model wardrop_tax.mod;
data wardrop_tax.dat;

problem NashGame:
    ### variables ... t[i,j] fixed automatically
    {(i,j) in ARCS} f[i,j],
    {(i,j) in ARCS, k in COMMODITIES} x[i,j,k],
    {i in NODES, k in COMMODITIES} p[i,k],
    ### constraints
    {(i,j) in ARCS, k in COMMODITIES} d_obj[i,j,k],
    {i in NODES, k in COMMODITIES} conserve[i,k],
    {(i,j) in ARCS} flow[i,j];

# ... solve NCP with fixed tolls
option solver pathampl; solve NashGame;
```
problem Stackelberg:

revenue, # objective

### variables ... added tolls t[i,j]
{(i,j) in TOLL} t[i,j], {(i,j) in ARCS} f[i,j],
{(i,j) in ARCS, k in COMMODITIES} x[i,j,k],
{i in NODES, k in COMMODITIES} p[i,k],

### constraints
{(i,j) in ARCS, k in COMMODITIES} d_obj[i,j,k],
{i in NODES, k in COMMODITIES} conserve[i,k],
{(i,j) in ARCS} flow[i,j];

option mpec_options "compl_ini=1"; # hot start
option solver mpec; solve Stackelberg;

display f; display tl, t, tu;
MORE EXAMPLES
More Examples of Stackelberg Games

- CO$_2$ abatement & transboundary pollution
- interaction of NO$_x$ & electricity markets
- volatility estimation in American option pricing
- transportation network design

MPECs allow us to ...

1. **model complex games** (heterogeneous players):
   e.g. electricity markets: ISO, generator

2. **simulate large & complex games**
   ... may need NCP approach first ...

3. **extend to design of markets**
   e.g. toll-pricing (where to raise tolls)
   ⇒ **integer variables** ... harder problem
FORMULATION OF MPECs AS NLPs
Mathematical Program with Equilibrium Constraints (MPEC)

\[
\begin{align*}
\text{minimize} & \quad f(x) & \quad \text{objective} \\
\text{subject to} & \quad c(x) \geq 0 & \quad \text{constraints} \\
& \quad 0 \leq x_1 \perp x_2 \geq 0 & \quad \text{complementarity}
\end{align*}
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where \( x = (x_0, x_1, x_2) \) ... partition of variables and

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0 \leq x_1 \perp x_2 \geq 0 \iff \text{either } x_{1i} = 0 \text{ or } x_{2i} = 0
\]

... equality constraints \( h(x) = 0 \) are no problem!
A Nonlinear Programming Approach

Replace equilibrium $0 \leq x_1 \perp x_2 \geq 0$ by $x_1^T x_2 \leq 0$

$\Rightarrow$ standard nonlinear program (NLP)

\[
\text{(NLP)} \begin{cases}
\text{minimize} & f(x) \\
\text{subject to} & c(x) \geq 0 \\
& x_1, x_2 \geq 0 \\
& X_1 x_2 \leq 0
\end{cases}
\]

Advantage: standard (?) NLP; use large-scale solvers ...

Snag: nonlinear program (NLP) violates standard assumptions!
Mangasarian Fromowitz CQ fails

Mangasarian Fromowitz Constraint Qualification at feasible $\hat{x}$:

\[ \hat{x}_1 = 0, \quad \hat{x}_2 > 0 \]

\[ \Rightarrow x_1 \geq 0, \quad \text{and} \quad x_2 x_1 \leq 0 \quad \text{active} \]

\[ \Rightarrow s_1 > 0, \quad \text{and} \quad \hat{x}_2 s_1 < 0 \]

MFCQ is minimalistic **stability assumption** for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set **unbounded**
Mangasarian Fromowitz CQ fails

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MFCQ is minimalistic stability assumption for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set unbounded
2. Constraint gradients linearly dependent
Mangasarian Fromowitz CQ fails

Mangasarian Fromowitz Constraint Qualification at feasible $\hat{x}$:

\[ \hat{x}_1 = 0, \hat{x}_2 > 0 \]

⇒ \( x_1 \geq 0, \) and \( x_2 x_1 \leq 0 \) active

⇒ \( s_1 > 0, \) and \( \hat{s}_2 s_1 < 0 \)

MFCQ is minimalistic stability assumption for NLP

Failure of MFCQ implies:

1. Lagrange multiplier set unbounded
2. Constraint gradients linearly dependent
3. Central path does not exist
Mangasarian Fromowitz CQ fails

Early (1980s) numerical experience disappointing:
  • failure on 60% of problems!!

Popular conclusion:
  • NLP approach is “inherently unstable”
  • must expect “failure in presence of round-off errors”
  • arbitrary small perturbation $\epsilon > 0$:
    $x_1^T x_2 \leq -\epsilon \Rightarrow \text{infeasible}$
Mangasarian Fromowitz CQ fails

Early (1980s) numerical experience disappointing:
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Popular conclusion:
- NLP approach is "inherently unstable"
- must expect "failure in presence of round-off errors"
- arbitrary small perturbation $\epsilon > 0$:
  $\Rightarrow x_1^T x_2 \leq -\epsilon \Rightarrow$ infeasible

Homer Simpson Approach to Numerical Analysis ...
- ... who perturbs structural zeros ???
- ... huge advances in NLP technology in 90s ...
CONVERGENCE OF NLP APPROACH
MPECs as NLPs

Mathematical Program with Equilibrium Constraints (MPEC)

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & c(x) \geq 0 \\
& 0 \leq x_1 \perp x_2 \geq 0
\end{align*}
\]

... formulated as nonlinear program (NLP):

\[
\begin{align*}
\text{minimize} \quad & f(x) \\
\text{subject to} \quad & c(x) \geq 0 \\
& x_1, x_2 \geq 0 \\
& X_1 x_2 \leq 0
\end{align*}
\]
Stationarity & Bounded Multipliers

Example \( x^* = (0, 1) \): 

\[
\begin{align*}
\min_{x} & \quad \frac{1}{2}(x_1 - 1)^2 + (x_2 - 1)^2 \\
\text{s.t.} & \quad x_1, x_2 \geq 0, \quad x_1 x_2 \leq 0
\end{align*}
\]

First order conditions:

\[
\begin{pmatrix}
-1 \\
0
\end{pmatrix} = \begin{pmatrix}
\nu_1 \\
0
\end{pmatrix} - \begin{pmatrix}
\xi \\
0
\end{pmatrix}
\]

\( \nu_1 \) multiplier of \( x_1 \geq 0 \); \( \xi \) multiplier of \( x_1 x_2 \leq 0 \).

Equivalent NLP \((x_1 x_2 \leq 0)\) violates MFCQ

\( \Rightarrow \) unbounded multipliers

Multipliers form a ray \( \Rightarrow \exists \) bounded multipliers
Optimality Conditions for MPECs

Optimality conditions of equivalent NLP: \( \exists \lambda^*, \nu_1^*, \nu_2^*, \xi^* \geq 0 \)

\[
\nabla f(x^*) - \nabla c(x^*)^T \lambda^* - \begin{pmatrix}
0 \\
\nu_1^* - X_2^* \xi^* \\
\nu_2^* - X_1^* \xi^*
\end{pmatrix} = 0 \quad 1^{st} \text{ order}
\]

\[c(x^*) \geq 0, \ x_1^* \geq 0, \ x_2^* \geq 0 \ \text{and} \ \ X_1^* x_2^* \leq 0 \quad \text{primal feas.}
\]

\[c(x^*)^T \lambda = x_1^*^T \nu_1^* = x_2^*^T \nu_2^* = 0 \quad \text{compl. slack.}
\]

\[\ldots \ \xi > 0 \ \text{allows} \ \nu_1^* - X_2^* \xi^* < 0 \ \ldots \ \text{equality constraint} \ x_1 = 0
\]

Multipliers bounded if \( \|\xi^*\| < \infty \)
Convergence of SQP Methods

SQP: quadratic/linear approximation of NLP [Todd’s talk]

**Theorem**: SQP converges quadratically!

**Proof**: ... paraphrased from 7 pages ...

1. if \( x_1^{(k)^T} x_2^{(k)} = 0 \) then \( x_1^{(k+1)^T} x_2^{(k+1)} = 0 \)
   \[ \Rightarrow \] standard SQP convergence proof

2. if \( x_1^{(k)^T} x_2^{(k)} > 0 \) then ...
   ... QP solver always picks non-singular basis
   \[ \Rightarrow \] standard SQP convergence proof
Do Slacks Matter ???

Consider general equilibrium $0 \leq G(x) \perp H(x) \geq 0$:
Do Slacks Matter ???

Consider general equilibrium $0 \leq G(x) \perp H(x) \geq 0$: MPEC with nonlinear complementarity condition

“nice” stationary points at $(0, 2)$ and $(2, 0)$

\[
\begin{align*}
\min_{x} & \quad -x_1 - \frac{1}{2}x_2 \\
\text{subject to} & \quad x_1 + x_2 \leq 2 \\
& \quad 0 \leq x_1^2 - x_1 \perp x_2 \geq 0 \quad \text{nonlinear}
\end{align*}
\]

$x_0 = (-\epsilon, t)^T$ for $t \geq 0$:
- $x_k \rightarrow (0, t)^T$ quadratically
- $\xi \rightarrow \infty$ weird !!!
- active set singular in limit

$\Rightarrow$ SQP gets stuck
DETOUR:
INTERIOR POINT METHODS
Interior Point Methods (IPM)

General NLP

$$\min_{x} f(x) \quad \text{subject to} \quad c(x) = 0 \quad \& \quad x \geq 0$$

Perturbed $\mu > 0$ optimality conditions $(x, z \geq 0)$

$$F_{\mu}(x, y, z) = \begin{cases} \nabla f(x) - \nabla c(x)^T y - z \\ c(x) \\ Xz - \mu e \end{cases} = 0$$

- Primal-dual formulation
- Central path $\{x(\mu), y(\mu), z(\mu) : \mu > 0\}$
- Apply Newton’s method for sequence $\mu \downarrow 0$
Newton’s method applied to primal-dual system gives ...

\[
\begin{bmatrix}
\nabla^2 \mathcal{L}_k & -A_k & -I \\
A_k^T & 0 & 0 \\
Z_k & 0 & X_k
\end{bmatrix}
\begin{bmatrix}
\Delta x \\
\Delta y \\
\Delta z
\end{bmatrix}
= -F_\mu(x_k, y_k, z_k)
\]

where \( A_k = \nabla c(x_k)^T \), \( X_k \) diagonal matrix of \( x_k \).

Solvers: LOQO, KNITRO, IPOPT ... different factorization
Interior Point Methods (IPM)

\[
\begin{align*}
\text{minimize} & \quad f(x) \quad \text{subject to} \quad c(x) = 0 \quad \& \quad x \geq 0 \\
\text{Related to barrier methods for handling inequalities} & \\
\begin{cases}
\text{minimize} & \quad f(x) - \mu \sum \log(x_i) \\
\text{subject to} & \quad c(x) = 0
\end{cases}
\end{align*}
\]

\[
\text{minimize} \quad x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 + x_2^2 \geq 1
\]
Interior Point Methods (IPM)

\[
\begin{align*}
\text{minimize} \quad & f(x) \quad \text{subject to} \quad c(x) = 0 \quad \& \quad x \geq 0 \\
\text{Related to barrier methods for handling inequalities} \\
\begin{cases}
\text{minimize} \quad & f(x) - \mu \sum \log(x_i) \\
\text{subject to} \quad & c(x) = 0
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\text{minimize} \quad & x_1^2 + x_2^2 \quad \text{subject to} \quad x_1 + x_2^2 \geq 1
\end{align*}
\]
BACK TO MPECs
Convergence of Interior Point Methods

**Key idea:** MPEC-stationarity $\iff 1^{st}$ order conditions of NLP

Approaches:
Convergence of Interior Point Methods

Key idea: MPEC-stationarity ⇔ 1st order conditions of NLP

Approaches:

1. Penalize equilibrium minimize $f(x) + \pi x_1^T x_2$
   - Well behaved smooth problem: constraints satisfy MFCQ
   - $\pi^* < \infty$ and $X_1 x_2 \leq 0 \Rightarrow$ strongly-stationary
   - How to adjust $\pi$? ... during barrier solve!
Key idea: MPEC-stationarity $\iff$ 1$^{st}$ order conditions of NLP

Approaches:

1. Penalize equilibrium minimize $f(x) + \pi x_1^T x_2$
   - Well behaved smooth problem: constraints satisfy MFCQ
   - $\pi^* < \infty$ and $X_1 x_2 \leq 0 \Rightarrow$ strongly-stationary
   - How to adjust $\pi$? ... during barrier solve!

2. Relax equilibrium $X_1 x_2 \leq \tau$ & $x_1, x_2 \geq -\delta$
   - Well behaved smooth problem
   - Adjust $\tau_i \downarrow 0$ or $\delta_i \downarrow 0$ ... not both
Convergence of Interior Point Methods

**Key idea:** MPEC-stationarity $\iff 1^{st}$ order conditions of NLP

**Approaches:**

1. **Penalize equilibrium** minimize $f(x) + \pi x_1^T x_2$
   - Well behaved smooth problem: constraints satisfy MFCQ
   - $\pi^* < \infty$ and $X_1 x_2 \leq 0 \Rightarrow$ strongly-stationary
   - How to adjust $\pi$? ... during barrier solve!

2. **Relax equilibrium** $X_1 x_2 \leq \tau$ & $x_1, x_2 \geq -\delta$
   - Well behaved smooth problem
   - Adjust $\tau_i \downarrow 0$ or $\delta_i \downarrow 0$ ... not both

**Aim:** NLP solver with small modification works for MPECs
Interior Point Method with Two Sided Relaxation

- MPECs have no strict interior
- Relaxation $X_1x_2 \leq \tau \downarrow 0 \Rightarrow \text{interior} \rightarrow 0$

$\Rightarrow \text{relax } X_1x_2 \leq \tau$
and $x_1 \geq -\delta, x_2 \geq -\delta$
... adjust $\tau, \delta$ as $\mu \rightarrow 0$

\textbf{Theorem}: In limit $\tau_i \rightarrow 0$ or $\delta_i \rightarrow 0$ but not both
$\Rightarrow$ relaxed problem has \textit{non-empty} interior in limit

MPEC multiplier $\mu_i < 0 \Rightarrow$ reduce $\tau_i \downarrow 0$ ...
Limitations of NLP Approach

- formulation of followers as NCP
  ⇒ questionable for nonconvex games
- MPEC-stationarity not necessarily no descent
- multipliers $\xi \to \infty$ ... not a solution???

... NLP approach remains most versatile MPEC solver
... allows simulation of heterogeneous leader-follower games
MULTI-LEADER-FOLLOWER GAMES
Multi-Leader-Follower Games

Non-cooperative equilibrium between $> 2$ dominant producers

\[
x_i^*, y \in \arg\min_{x_i \geq 0, y} f_i(x, y)
\text{s.t.} \quad h(x_i, y) = 0 \quad 0 \leq y_1 \perp y_2 \geq 0
\]

Complementarity: optimality conditions of leaders
⇒ equilibrium problem with equilibrium constrains (EPEC)

Leaders’ MPECs violate Mangasarian-Fromowitz CQ
... are EPECs harder than MPECs ???

... recall MPEC strong-stationarity ⇒ ∃ bounded multipliers
Initial experience: no feasible solution to MLF game

Equilibrium may not exist:

\[
\begin{cases}
\text{minimize} & \begin{pmatrix}
\frac{1}{2}x_1 + z \\
-\frac{1}{2}x_2 + z
\end{pmatrix} \\
\text{subject to} & z - 1 + x_1 - x_2 - \lambda = 0 \\
& 0 \leq z \perp \lambda \geq 0
\end{cases}
\]

\[\Rightarrow z(x_1, x_2) = \max(0, 1 - x_1 - x_2)\]
A Practical Nonlinear Approach to EPECs

Initial experience: no feasible solution to MLF game

Equilibrium may not exist:

\[
\begin{align*}
\text{minimize} & \quad \left( \begin{array}{c}
\frac{1}{2}x_1 + z \\
-\frac{1}{2}x_2 + z
\end{array} \right) \\
\text{subject to} & \quad z - 1 + x_1 - x_2 - \lambda = 0 \\
& \quad 0 \leq z \perp \lambda \geq 0
\end{align*}
\]

\[\Rightarrow z(x_1, x_2) = \max(0, 1 - x_1 - x_2)\]

... suggest convexification approach to get feasible ...
Gauss-Seidel Iteration for EPECs

set I := 1..2;

var x{I};   # ... leader variables
var s;       # ... follower variables
var y;

# ... leader’s objective
minimize leader{i in I}: (x[i]+1)^2;

subject to
  compl: 0 <= s complements y >= 0;
Gauss-Seidel Iteration for EPECs

\[
\begin{align*}
\text{param } x0\{I\} & \text{ default } 0; \quad \# \text{ iter } k \\
\text{param } s0\{I\} & \text{ default } 0; \\
\text{param } y0\{I\} & \text{ default } 0; \\
\text{param } x1\{I\}; \text{ param } s1\{I\}; \text{ param } y1\{I\}; \quad \# \text{ iter } k+1 \\
\text{param } Tol & \geq 0, \text{ default } 1E-4; \quad \# \text{ tolerance} \\
\text{param } MaxIt & \geq 0, \text{ integer, default } 20; \quad \# \text{ max iter} \\
\text{param } Err & \text{ default } 0; \\
\text{let} \{i \in I\} x[i] & := x0[i]; \quad \# \text{ init vars} \\
\text{let } s & := s0[1]; \text{ let } y & := y0[1]; \\
\text{problem Stackelberg}\{i \in I\}: \quad \# \text{ leader } i \\
\quad \text{leader}[i], \quad \# \text{ objective} \\
\quad x[i], y, s, \quad \# \text{ variables} \\
\quad \text{slack, compl}; \quad \# \text{ constraints}
\end{align*}
\]
Gauss-Seidel Iteration for EPECs

for {k in 1..MaxIt} {
    for {i in I} {
        solve Stackelberg[i];
        let x1[i] := x[i];
        let s1[i] := s;
        let y1[i] := y;
    };

    let Err := sqrt( sum{i in I}( (x0[i] - x1[i])^2 
                              + (y0[i] - y1[i])^2 
                              + (s0[i] - s1[i])^2 ));

    if (Err <= Tol) then break;

    let{i in I} x0[i] := x1[i];
    let{i in I} s0[i] := s1[i];
    let{i in I} y0[i] := y1[i];
};
Complementarity System Approach

Non-cooperative Nash equilibrium between leaders
\((\nabla_i = \nabla_{(x_i, y_i)})\) ... for all leaders \(i = 1, \ldots, L\):

\[
\nabla_i f_i(x, y) - \nabla_i h(x_i, y)\mu_i - \begin{pmatrix}
\sigma_i \\
0 \\
\nu_{i1} - \xi_i y_2 \\
\nu_{i2} - \xi_i y_1 \\
h(x_i, y)
\end{pmatrix} = 0
\]

\[
0 \leq x_i \perp \sigma_i \geq 0 , \quad 0 \geq y_1^T y_2 \perp \xi_i \geq 0 \\
0 \leq y_1 \perp \nu_{i1} \geq 0 , \quad 0 \leq y_2 \perp \nu_{i2} \geq 0
\]

Non-square complementarity problem
... OK for nonlinear optimizers
... note each leader has own multipliers \(\xi_i, \nu_i\)
... price-consistent formulation \(\xi_i = \xi \; \forall i \Rightarrow\) square NCP
Complementarity System Approach

var lambda{I}; var sigma{I};  # ... multipliers
var nu{I};    var xi{I};

subject to
   KKTx{i in I}: 2*(x[i]+1) + lambda[i] = 0;
   KKTy{i in I}: lambda[i] - nu[i] + xi[i]*s = 0;
   KKTs{i in I}: - lambda[i] - sigma[i] + xi[i]*y = 0;

slack{i in I}: s = x[1]+x[2]+y complements lambda[i];

bnds{i in I}:  0 <= s    complements sigma[i] >= 0;
 bndy{i in I}:  0 <= y    complements nu[i]  >= 0;
 compl{i in I}: 0 <= -y*s complements xi[i]  >= 0;
Solution of Multi-Leader Follower Games

- MPECs are nonconvex
  ⇒ first-order conditions are not sufficient
  ⇒ NCP approach may fail
- Differentiation is not easy
  ⇒ mistakes in NCP formulation
- MPECs are already difficult
  ⇒ Gauss-Seidel often fails
- penalized approach works best at present
CONCLUSIONS
Conclusions

- complex leader follower games are MPECs/EPECs
- numerical treatment of large problems possible

⇒ can simulate complex games numerically
Conclusion & Outlook

Conclusions

• complex leader follower games are MPECs/EPECs
• numerical treatment of large problems possible

⇒ can simulate complex games numerically

Open Questions

• Interpretation of NLP failures?
• Can we avoid “spurious-stationary” points?
• How to get global minimizers for some applications?
“Homework”

1. Formulate hakonsen.mod as an NLP; replacing $0 \leq x_1 \perp x_2 \geq 0$ by
   - 1.1 $\min(x_1, x_2) \leq 0$ and $x_1, x_2 \geq 0$
   - 1.2 $x_1^T x_2 \leq 0$ and $x_1, x_2 \geq 0$
   ... and solve the model using snopt, loqo et al.

2. Solve wardop.mod/wardrop-01.dat:
   - 2.1 starting from default
   - 2.2 starting from point given in wardrop-01.start

   What is the effect on the solvers? Compare IPM to ASM!

Models: www.mcs.anl.gov/~leyffer/ice05/