# A Model of Traffic Congestion, Housing Prices and Compensating Wage Differentials

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**Outline**

1. Literature review and motivation.

2. Computing Wardropian equilibria in a complementarity framework.

3. A computable spatial equilibrium model of employment demand and regional housing markets.

4. Some illustrative calculations.

## This Paper's Contribution

This paper illustrates how the Wadropian equilibrium concept can be embedded in a Walrasian, general equilibrium model in which the response of housing rental rates, wages, employment and population density can be evaluated in response to changes to highway capacity.

**Small (1997)** "Economics and Urban Transportation Policy in the United States" *Regional Science and Urban Economics*.

What role can/should economics can play in analysing problems with urban transportation in the United States?

Five specific problems: infrastructure funding, financially weak public transit, environmental externalities, motor vehicle accidents and traffic congestion.

## Social Costs of Urban Automobile Travel

| Type of Cost | Cost ($ / vehicle-miles) |
|---|---|
| (1) Running cost | 0.075 |
| (2) Vehicle capital | 0.204 |
| (3) Time | 0.152 |
| (4) Schedule delay | 0.083 |
| (5) Accidents | 0.110 |
| (6) Parking (CBD fringe) | 0.150 |
| *Total:* | |
| Without parking | 0.624 |
| With Parking | 0.774 |

Global warming with carbon taxes on the order of \$50 per ton carbon-dioxide correspond to \$0.1 per gallon of gasoline or \$0.01 per vehicle-mile.

**Vickrey (1969)** "Congestion Theory and Transport Investment", *American Economic Review Papers and Proceedings*.

"... traffic often behaves like population. It has been said that if nothing stops the growth of population but misery and starvation, then the population will grow until it is miserable and starves."

## Various Types of Congestion

i. *Simple interaction*. Occurs whenever two transportation units approach each other such that one must delay to reduce the likelihood of a collision. (Light traffic) In this case, congestion delay tends to vary as the square of the volume of traffic. A motorist deciding on a trip will thereby inflict on others an amount of additional delay roughly equal to what he himself will experience.

ii. *Multiple interaction*. This occurs at higher levels of traffic density, short of capacity flows. In this case, average speed $s$ is a function of the flow of traffic $x$:

$$s = f(x)$$

For traffic volumes ranging from 0.5 to 0.9 of capacity, one can fit a function of the form:

$$z = t - t_0 = \frac{1}{s} - \frac{1}{s_0} = ax^k$$

where $t$ is the time of travel under actual conditions, $t_0$ is the time required with no traffic, $z$ is delay per vehicle, and $a$ and $k$ are constants.

Total increment of delay:

$$\frac{d(zx)}{dx} = z + x\frac{dz}{dx} = ax^k + xakx^{k-1} = (1+k)z$$

This the delay that results from a unit increment of traffic works out to $k+1$ times the delay experienced by the incremental vehicle. For every minute that is experienced by that vehicle, $k$ minutes of delay are inflicted on the remaining traffic.

For situations where considerable congestion exists, $k$ is likely to be in the range of 3 to 5 or higher. The simple interaction case involves $k = 1$.

## Wardropian Equilibria

Wardrop, J. G. (1952), 'Some theoretical aspects of road traffic research', *Proceeding of the Institute of Civil Engineers, Part II* pp. 325–378.

Key idea: people aren't stupid. Drivers take the shortest route, taking decisions of other drivers as given.

A *multicommodity formulation* of Wardrop's model can provide a compact and efficient representation of the model, permitting direct solution with "off-the-shelf" algorithms. (Ferris, Meeraus and Rutherford, 1999.)

**Multicommodity Formulation**

A road network is defined by set of nodes ($\mathcal{N}$) and a set of directed arcs ($\mathcal{A}$).

The cost of traveling along a given arc is an increasing (nonlinear) function of the total flow along that arc, $c_{(i,j)}(f_{(i,j)})$.

Subsets of $\mathcal{N}$ include origin nodes ($\mathcal{O}$) and destination nodes ($\mathcal{D}$).

**Three Classes of Variables**

1. "Commodity flows", which are identified by the destination node ($k$) and the arc on which the flow occurs ($(i,j)$):

$x^k_{(i,j)}$ = flow of vehicles destine to node $k$ on arc $(i,j)$.

2. Arc flows, represent the total number of vehicles on a particular arc:

$f_{(i,j)} = \sum_{k \in \mathcal{D}} x^k_{(i,j)} \qquad \forall (i,j) \in \mathcal{A}$

3. Travel time from node $i$ to destination node $k$:

$t^k_i$ = the minimum time to reach node $k$ from node $i$.

**Wadropian Equilibrium**

1. Conservation of flow destine to $k$ at node $i$:

$$\sum_{j:(i,j)\in\mathcal{A}} x^k_{(i,j)} - \sum_{j:(j,i)\in\mathcal{A}} x^k_{(j,i)} = d^k_i \qquad \forall i \in \mathcal{N}, k \in \mathcal{D} \tag{1}$$

2. Rationality of flow along arc $(i,j)$ – a driver destine to node $k$ who originates at $i$ and travels to $j$ only takes that route if no shorter route exists:

$$c_{(i,j)}\left(f_{(i,j)}\right) + t^k_j \geq t^k_i \perp x^k_{(i,j)} \geq 0 \qquad \forall (i,j) \in \mathcal{A}, k \in \mathcal{D} \tag{2}$$

## Nonlinear Programming Formulations

If cost functions $c_a$ are integrable as $C_a$ for each $a \in \mathcal{A}$, then the equilibrium conditions are the first order optimality conditions of the nonlinear program:

$$
\begin{array}{lll}
\min_x & \sum_{a\in\mathcal{A}} C_a(F_a) & \\
\text{subject to} & Ax^k = d^k, x^k \geq 0 & \forall k \in \mathcal{D} \\
 & F_a = \sum_{\ell\in\mathcal{D}} x_a^\ell &
\end{array} \tag{3}
$$

An exact dual formulation is also possible.

**Model Specification**

Consider a road network ($\mathcal{N}$) described by the following data:

$L_{ij}$ the length of road $ij$ in the network,

$\ell_{ij}$ the number of lanes in road $ij$ in the network,

$s_{ij}$ the speed limit on road $ij$ (miles per hour).

Calibrating the model involves deriving parameters $\alpha$ and $\beta$ for the following congestion function:

$$\tau_{ij}(F) = \alpha_{ij} + \beta_{ij} F^4$$

Hence:

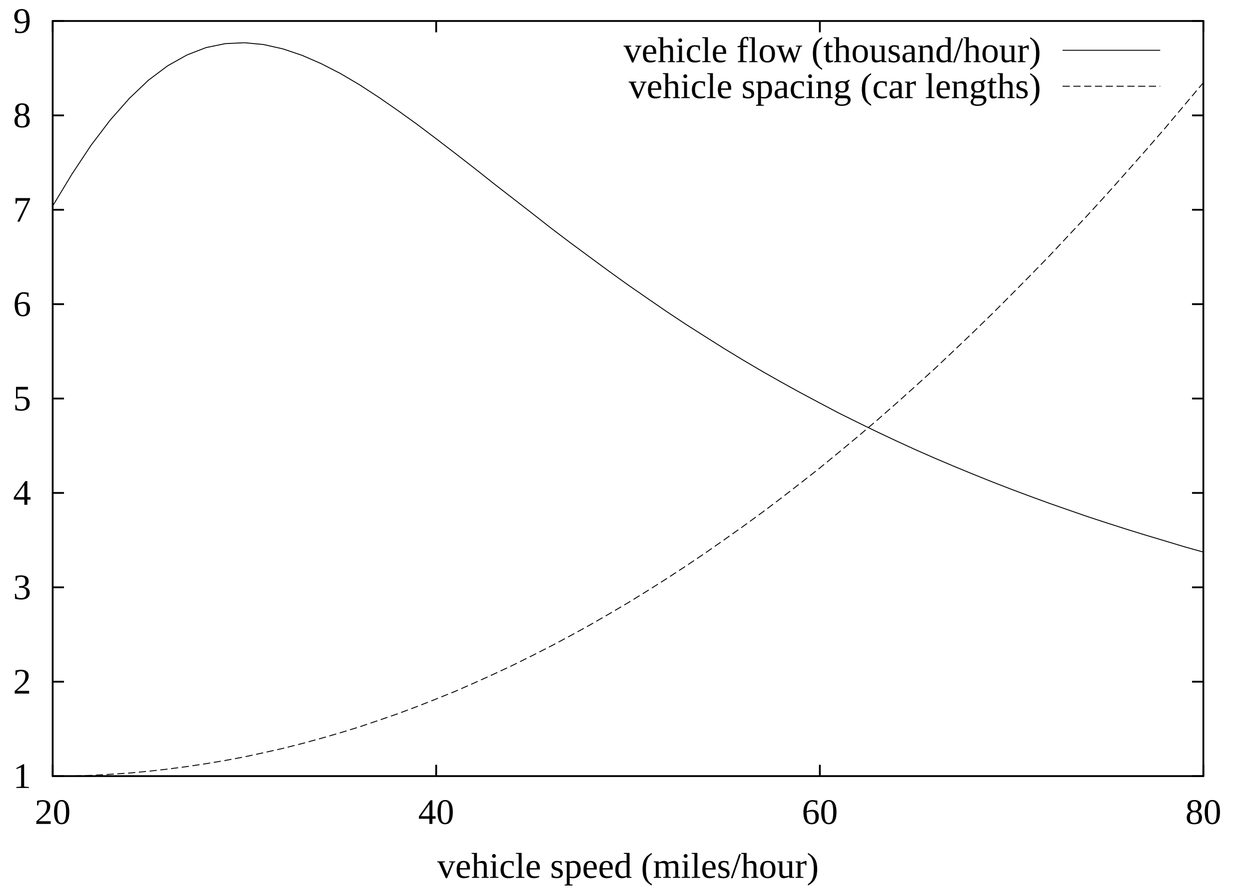
$$\alpha_{ij} = \frac{60 \times L_{ij}}{s_{ij}}$$

The $\beta$ parameter is less easily obtained from "facts" and perhaps less precisely specified.

When vehicles are driven on a highway with $\ell$ lanes, at speed $s$ (miles per hour) and are uniformly spaced at $d$ feet, the associated flow (in thousands of vehicles) is

$$F(\ell, s, d) = \frac{5.28 \times \ell \times s}{d}.$$

## Calibration from Quadratic Spacing



## MPEC Estimation: Numerical Calibration

An alternative to a survey-based estimates of $\bar{F}_{ij}$ for each arc would be to solve an inverse problem which seeks to solve

$$\min_{\beta} ||F(\beta) - \bar{F}||$$

## A Spatial Equilibrium Model

Our current research program seeks to formulate a spatial equilibrium model of regional housing markets which accounts for characteristics of roads, the housing stock and employment demand.

Our objective is to produce model which can be used to study the geography of a major metropolitan area through the representation of the locations for employment, housing and the connecting transportation arteries.

We will produce a model which can be used to study the interplay between the road system, the pattern and level of employment and the pattern and value of the housing stock.

## Notation

$i, j, k \in \mathcal{N}$ are indices which will be used to describe "nodes" in the network. Each node has associated employment and housing stock. In a small-scale network, each node might represent a specific intersection. In a larger-scale application, a node might represent a major interchange in a freeway system.

$a_{ij}$ denotes "transportation arcs" in the network. These arcs correspond to specific roads or major arteries in the road system.

$c_a(F_a)$ denotes the congestion function on arc $a$. Following the logic of Vickery's multiple interaction model, the time to traverse an arc is a function of the number of cars on that arc as follows:

$$c_a(F_a) = \alpha_a + \beta_a F_a^4$$

## Logic of the Economic Model

$U(c, T, H)$ denotes househould utility which depends on consumption, travel time and housing. In our illustrative calculations we represent this with the following parametric form:

$$U(c, T, H) = \left(\gamma(\bar{T} - T)^{\rho} + (1 - \gamma)\left(c^{\theta} H^{1-\theta}\right)^{\rho}\right)^{1/\rho}$$

$w_j \geq 0$ represents the wage paid by the employer at location $j$. This includes a premium which compensates for travel cost.

$D_j(w_j)$ represents the labor demand function by employers at node $j$, given by:

$$D_j(w_j) = \phi_j w_j^{-\sigma_j}$$

The equilibrium travel time from location $i$ to $j$ is $T_{ij}$.

A household is willing to live in location $i$ and commute to $j$ if the wage is sufficient to compensate for the loss of leisure required for the commute to work.

Budget-constrained utility maximization allocates income between goods and housing:

$$\max \quad U(c, T_{ij}, H) \quad \text{s.t.} \quad c + p_i^H H = w_j$$

Note that we have formulated this model as though all individuals are renters. The price of housing together with consumption then exhaust wage income.

Associated demand functions per household who lives at $i$ and works at $j$ may then be computed as functions of the housing price and the wage:

$$c_{ij}(w_j) = \theta w_j$$

and

$$H_{ij}\left(w_j, p_i^H\right) = (1-\theta)\frac{w_j}{p_i^H}$$

## Equilibrium Sorting

Substituting for $c$, $T$ and $H$ in the direct utility function then yields an indirect utility function, $V_{ij}(w_j, p_i^H, T_{ij})$.

The following arbitrage condition determines the number of households living at location $i$ and working at location $j$:

$$\hat{U} \geq V_{ij}(w_j, p_i^H, T_{ij}) \quad \perp \quad N_{ij} \geq 0$$

**N.B.** $N_{ij}$ will be zero whenever $V_{ij} < \hat{U}$.

In equilibrium the utility level of all households are equal to $\widehat{U}$. The number of households living at $i$ and working at $j$ is given by $N_{ij}$. This number is greater then zero only in the event that the realized utility level offered by that location is sufficient to entice households to locate there.

In equilibrium the housing market is cleared as:

$$\sum_j N_{ij} H_{ij}(w_j, p_i^H) = \bar{H}_i$$

where $\bar{H}_i$ is the housing stock at node $i$.

The model is then closed by assuming a fixed aggregate population of workers:

$$\bar{N} = \sum_{i,j} N_{ij} \quad \perp \quad \widehat{U}.$$

Alternatively could close the model by assuming that aggregate population is a function of the equilibrium utility level:

$$\psi \widehat{U}^{\mu} = \sum_{i,j} N_{ij}$$

## Multicommodity Formulation of Wardrop's Model

Travel time from node $i$ to $j$ satisfies the following arbitrage condition:

$$T_{ik} \leq c_{ij}(F_{ij}) + T_{jk} \quad \perp x_{ij}^{k}$$

*i.e.*, commuting time from location $i$ to location $k$ must be no greater than the travel time from $i$ to $j$ plus the travel time from $j$ to $k$.

**N.B.** Whenever $c_{ij}(F_{ij}) + T_{jk} > T_{ik}$ it immediately flows that $x_{ij}^{k} = 0$.

Flow conservation at location $j$ for persons commuting to work at location $k$ is given by:

$$N_{jk} + \sum_i x_{ij}^k = \sum_i x_{ji}^k \quad j \neq k \qquad \perp T_{jk}$$

and

$$N_{jj} + \sum_i x_{ij}^j = D_j(w_j) \qquad \perp w_j$$

## An Illustrative Example

```
set Blocks := 1..6;                                # Spatial discretization
set Nodes := 1..card(Blocks)*card(Blocks);         # Nodes in the network
set Arcs within Nodes cross Nodes;                 # Arcs in the network
check: 0 == card {i in Nodes: (i,i) in Arcs};

param xcoord{Nodes};
param ycoord{Nodes};

param gamma default 1.00;                  # Disutility of travel time
param theta default 0.66;                  # Consumption value share

param phi {Nodes} default 1.0;             # Labor demand coefficient
param h0 {Nodes} default (1.0-theta);      # Housing stock
param sigma {Nodes} default 0.5;           # Labor demand elasticity
param alpha {Arcs} default 0.1;            # Travel distance parameter
param beta {Arcs} default 1.0;             # Congestion parameter
```

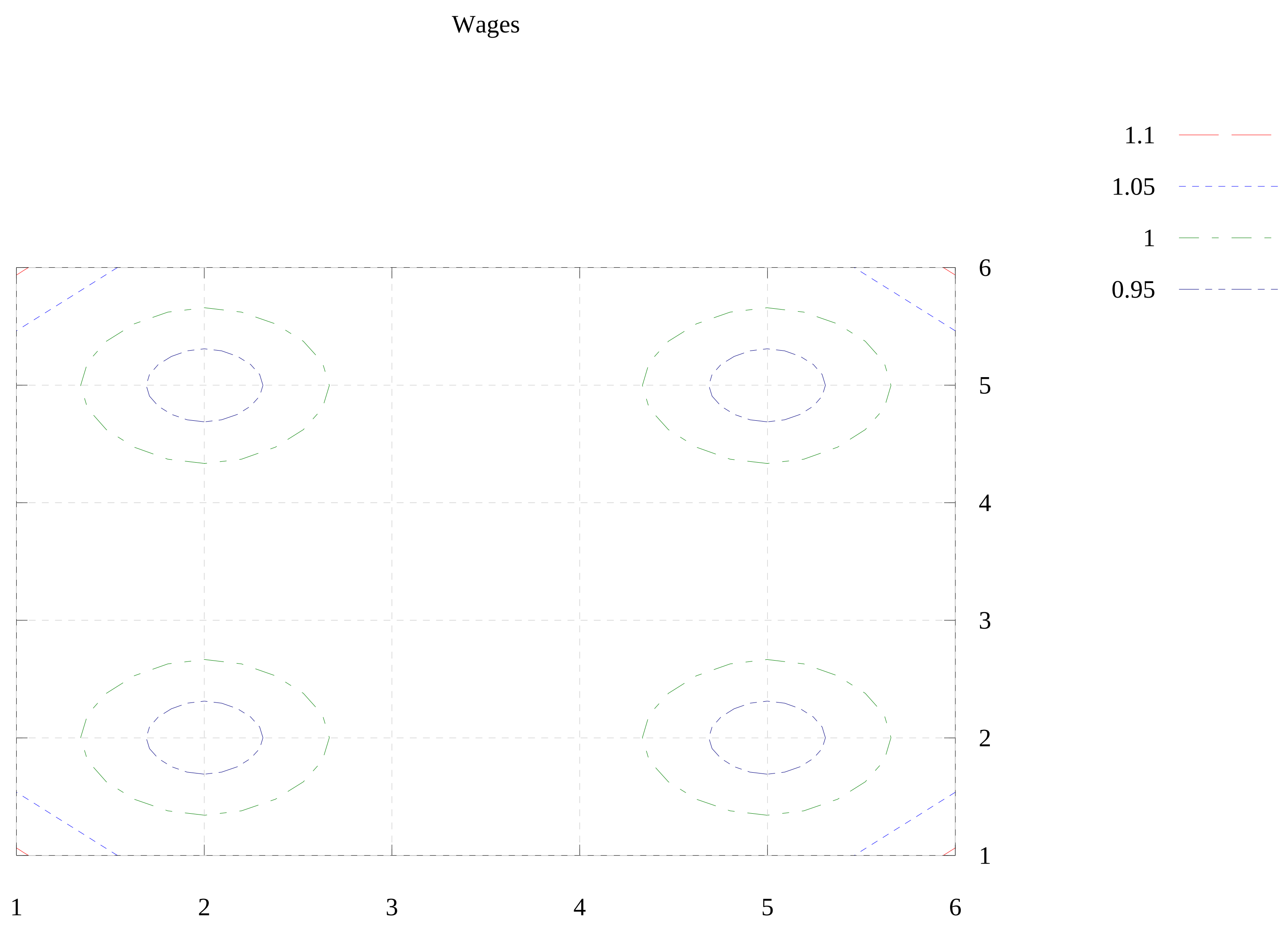
## Variables and Equations

```
var PH {Nodes};                             # Housing price
var N {Nodes, j in Nodes: phi[j] > 0};      # Number living at i and working at j
var W {j in Nodes: phi[j] > 0};             # Wage rate
var T {Nodes, j in Nodes: phi[j] > 0};      # Travel time
var F {Arcs};                               # Total Flow
var X {Arcs, j in Nodes: phi[j] > 0};       # Multicommodity flow
var U;                                      # Equilibrium utility
```

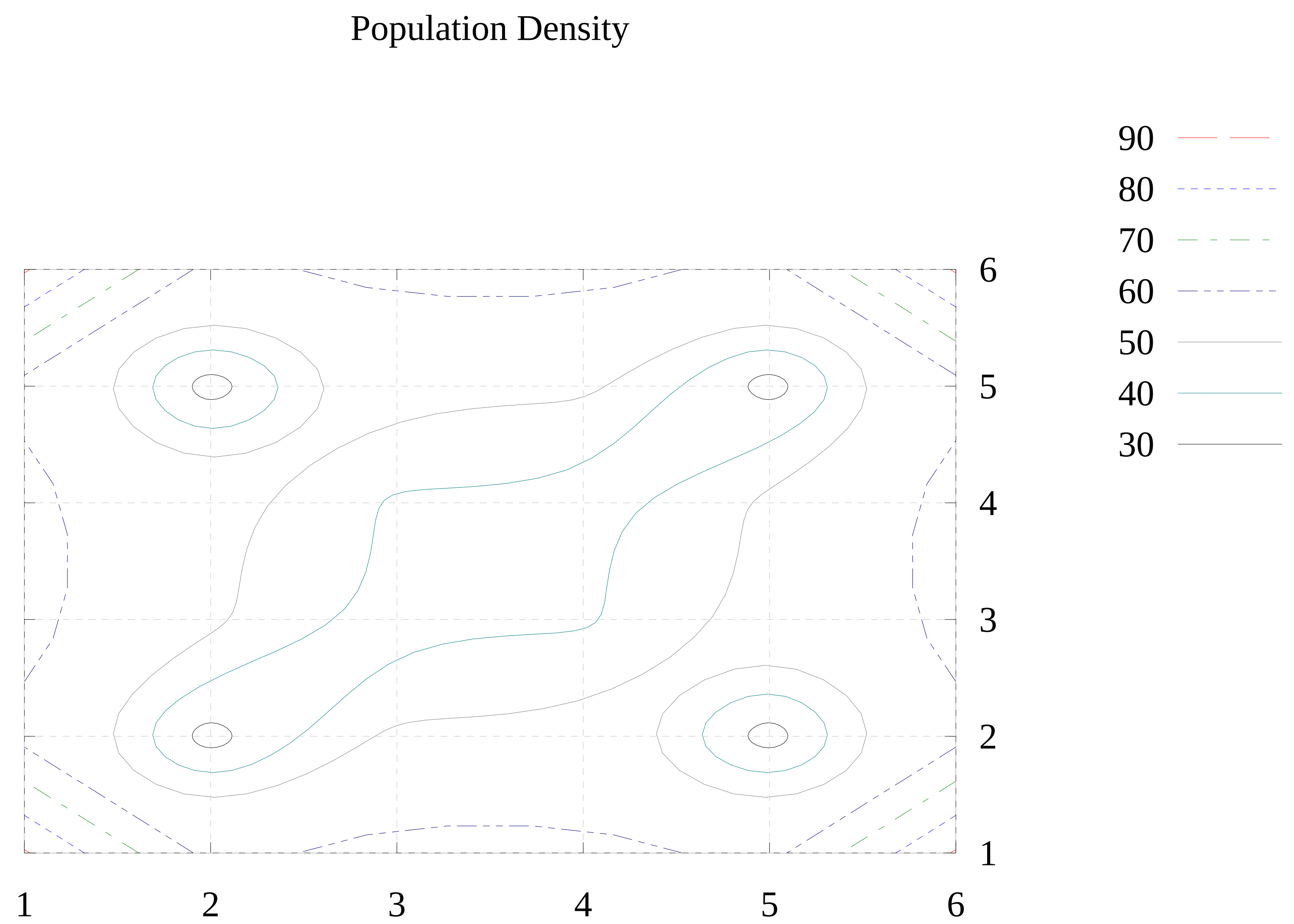
```
subject to
  housing{i in Nodes}:
    0.001 <= PH[i] complements
      h0[i]*PH[i] >= sum{j in Nodes: phi[j] > 0} (1.0-theta)*N[i,j]*W[j];

  number{i in Nodes, j in Nodes: phi[j] > 0}:
    0 <= N[i,j] complements
      U * PH[i]^(1.0-theta) >= exp(-gamma * T[i,j]) * W[j];

  balance{i in Nodes, k in Nodes: i <> k and phi[k] > 0}:
    0 <= T[i,k] complements
      sum{(i,j) in Arcs} X[i,j,k] >= N[i,k] + sum{(j,i) in Arcs} X[j,i,k];

  employment{j in Nodes: phi[j] > 0}:
    0.001 <= W[j] complements
      (N[j,j] + sum{(i,j) in Arcs} X[i,j,j])*W[j]^sigma[j] >= phi[j];

  time{(i,j) in Arcs, k in Nodes: phi[k] > 0}:
    0 <= X[i,j,k] complements
      T[j,k] + alpha[i,j] + beta[i,j]*F[i,j]^4 >= T[i,k];
```

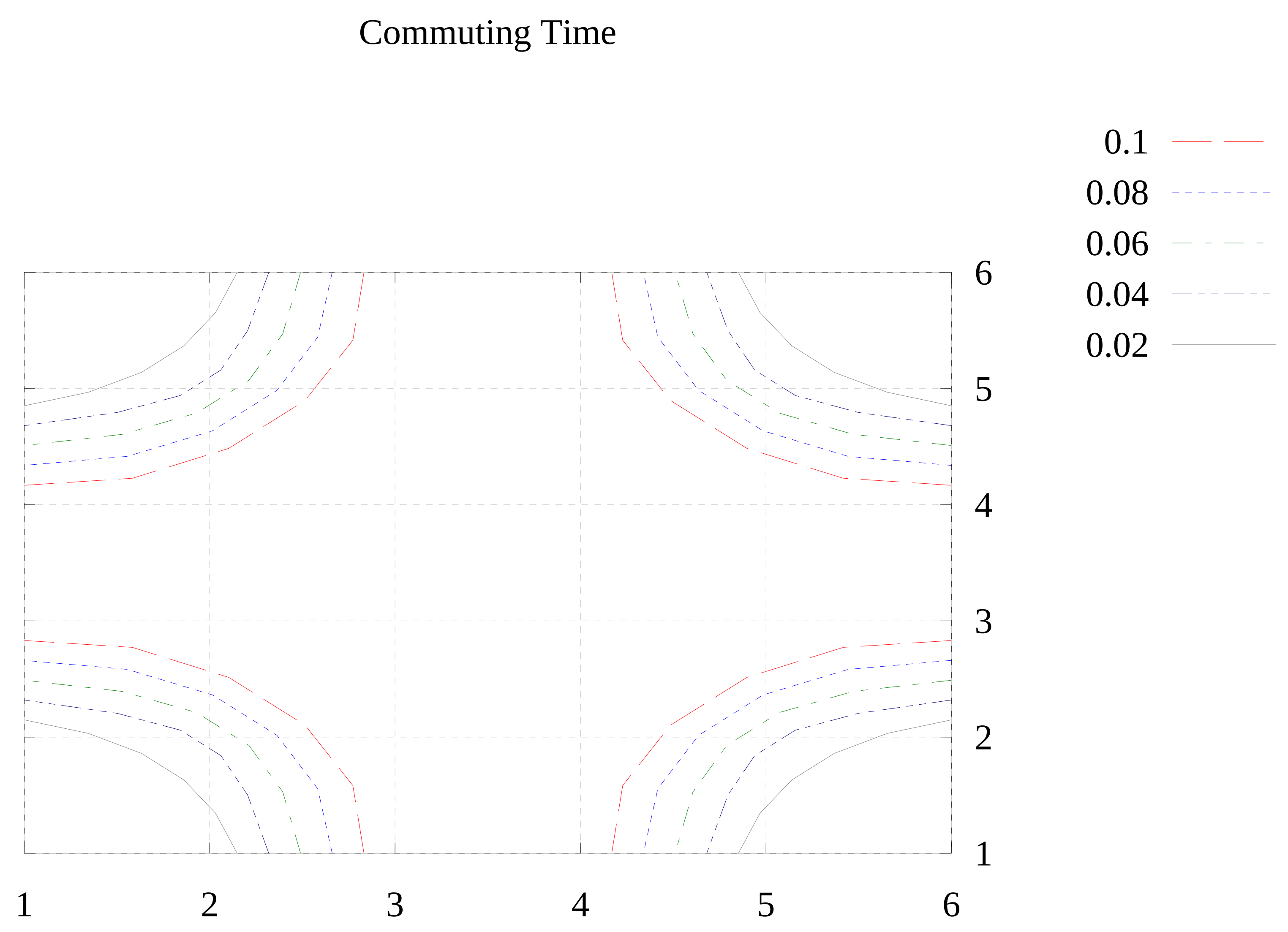
```
flow{(i,j) in Arcs}:
  F[i,j] complements
    F[i,j] = sum{k in Nodes: phi[k] > 0} X[i,j,k];

population:
  U complements
    card(Nodes) = sum{i in Nodes, j in Nodes: phi[j] > 0} N[i,j];
```

*Example 1: Jobs are located in the corners*

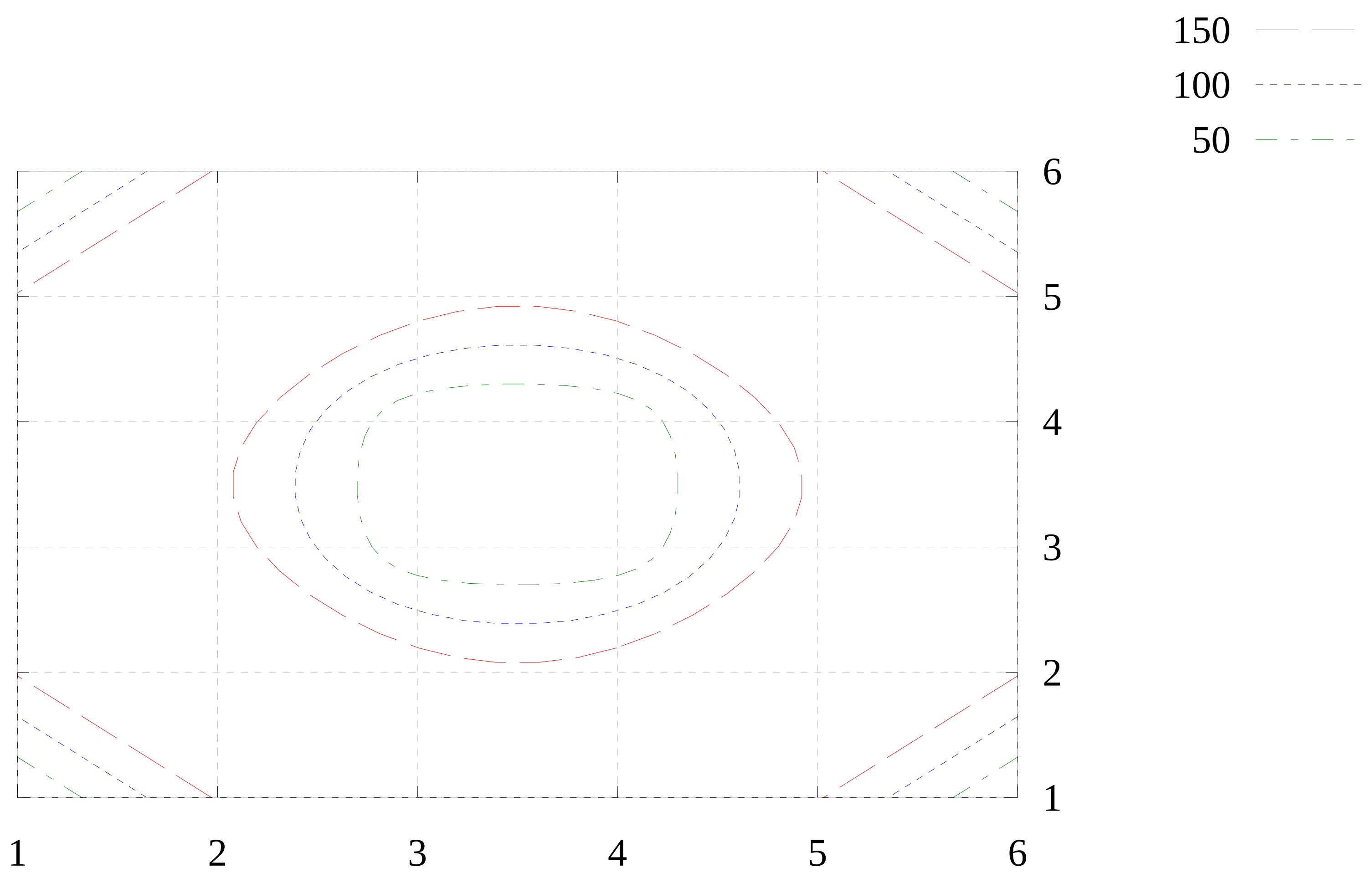

Wages
1.1
1.05
1
0.95
1
2
3
4
5
6
1
2
3
4
5
6

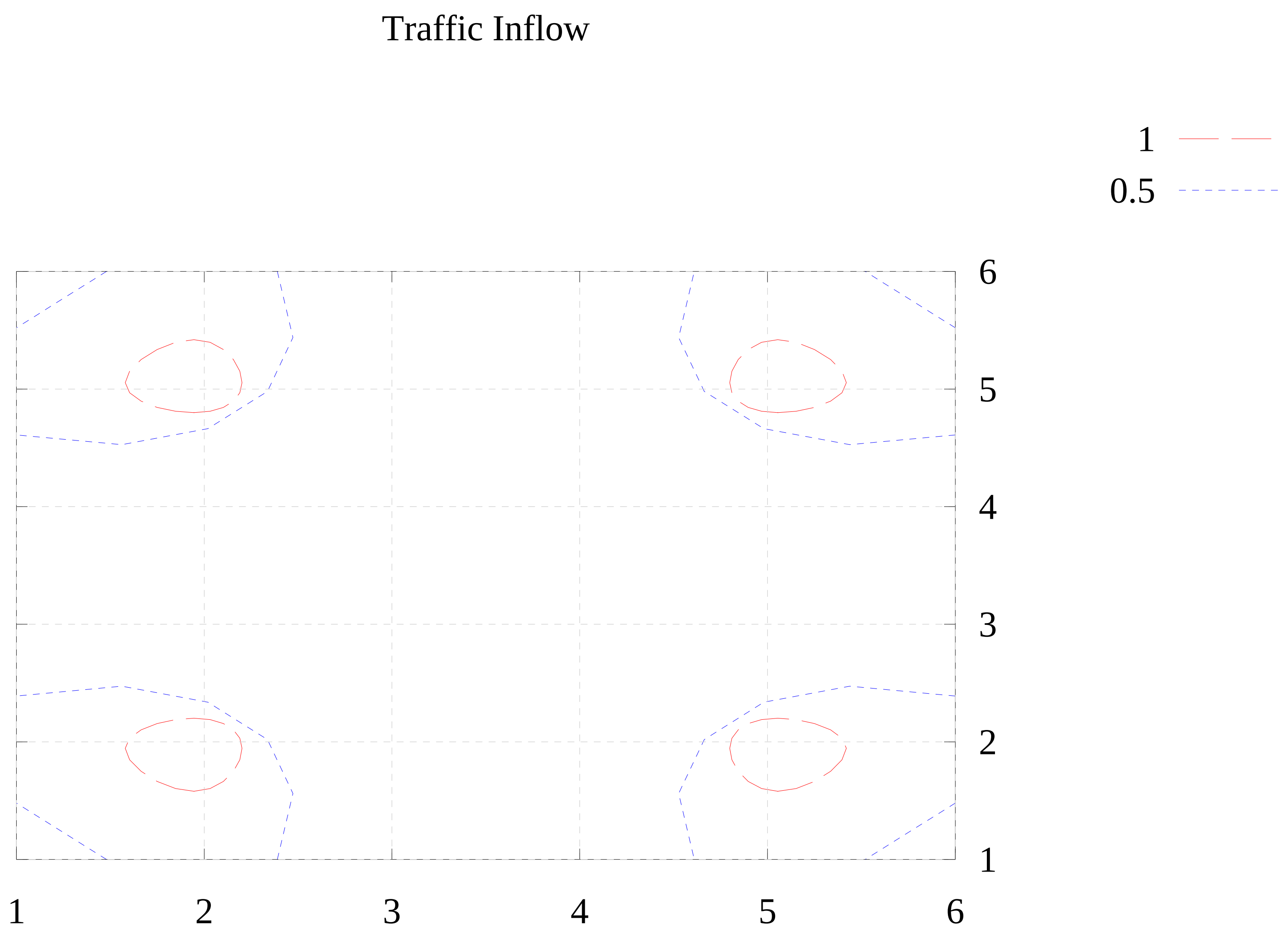
Housing Prices
2
1.5
1
0.5
6
5
4
3
2
1
1
2
3
4
5
6

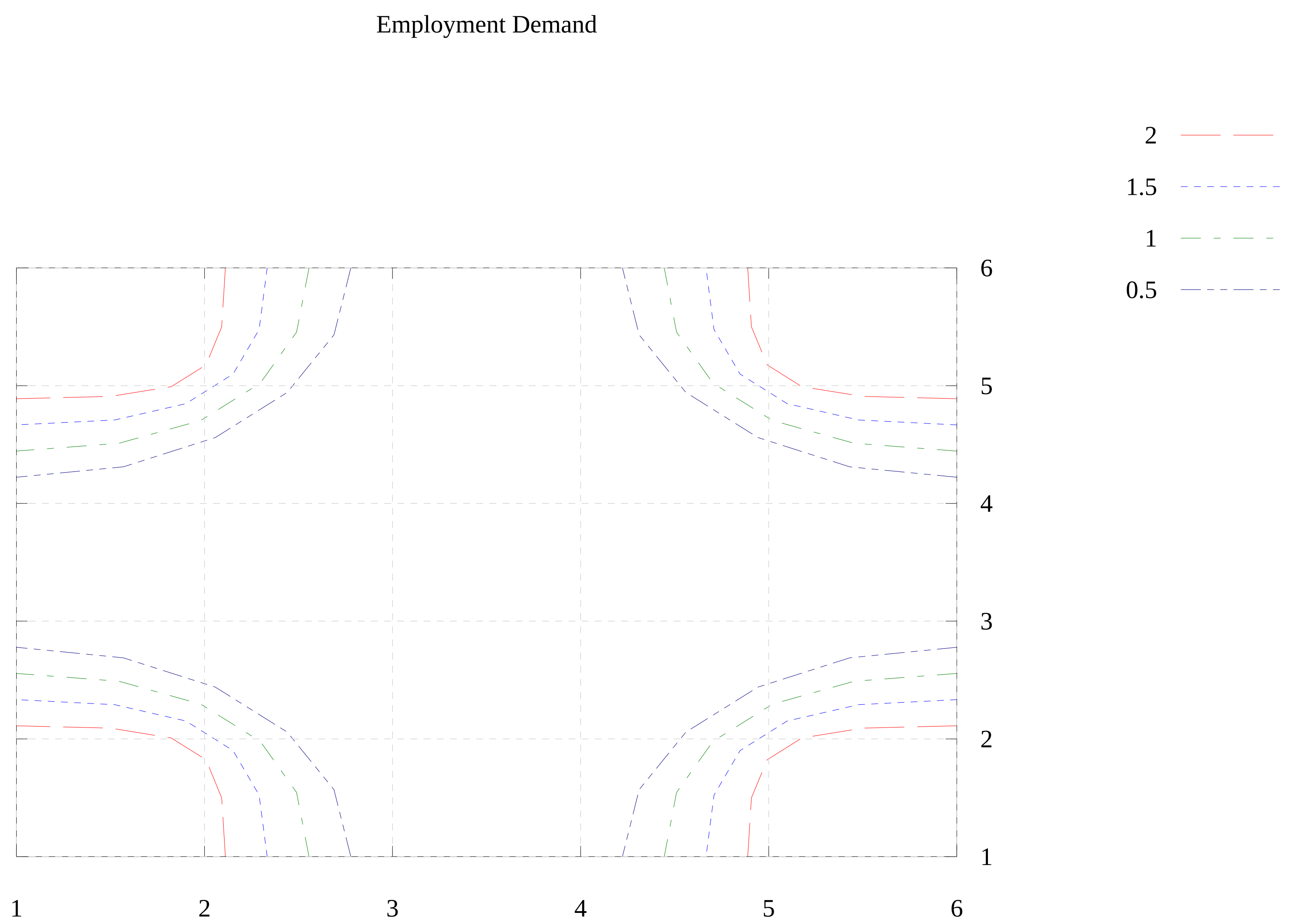

Population Density
90
80
70
60
50
40
30
1
2
3
4
5
6
1
2
3
4
5
6


Commuting Time
0.1
0.08
0.06
0.04
0.02
1
2
3
4
5
6
1
2
3
4
5
6

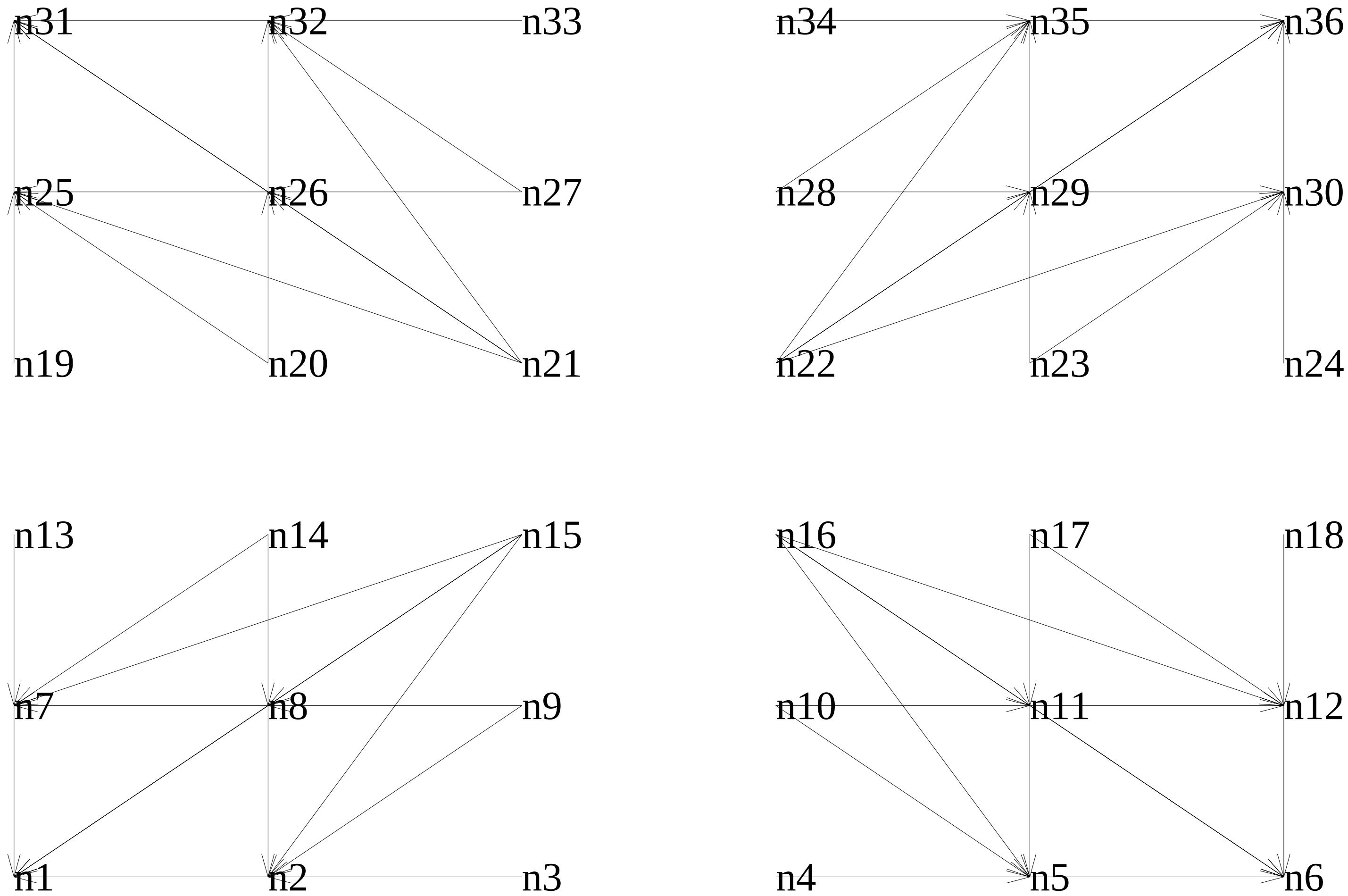
Average Congestion



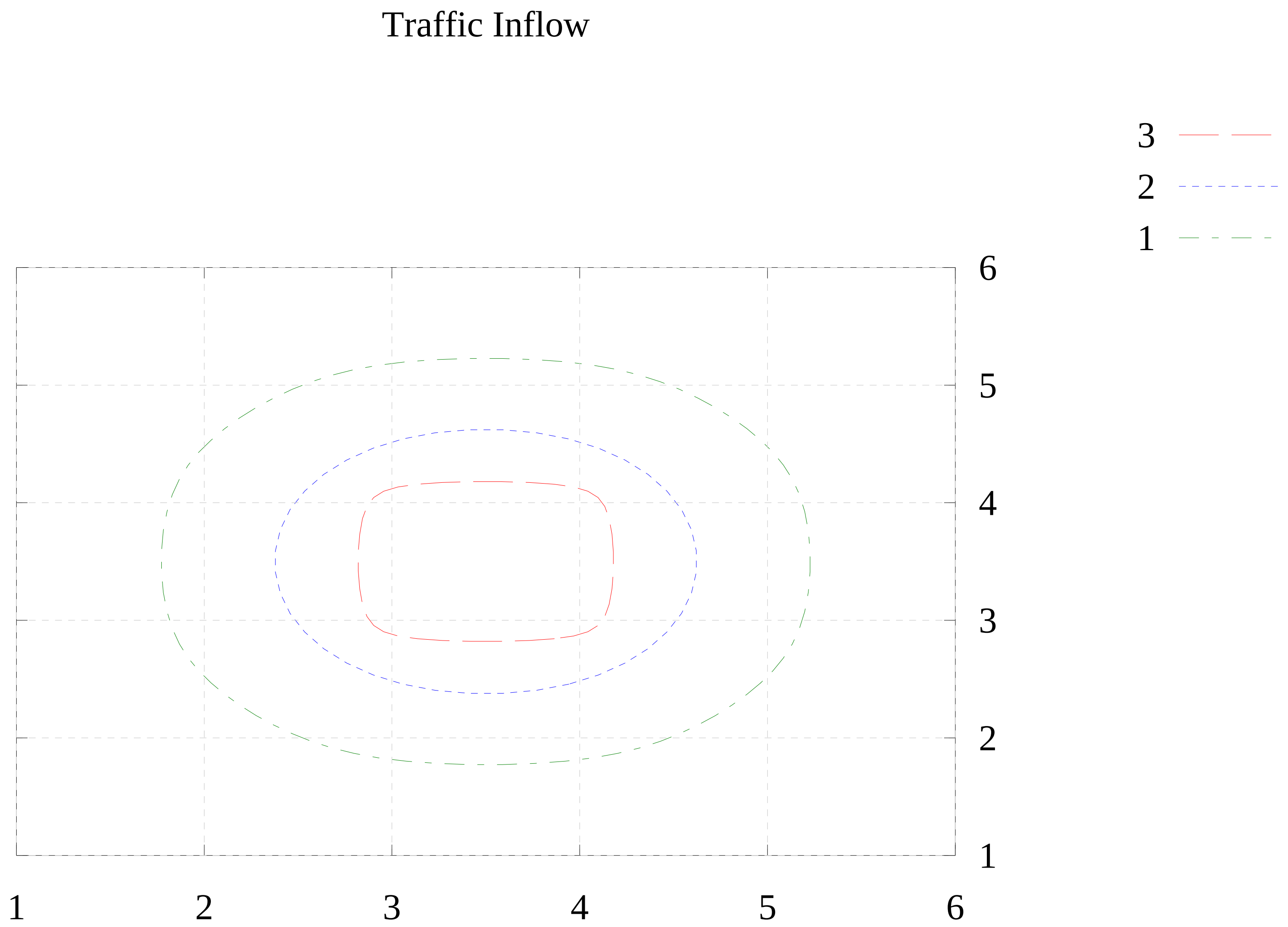

Traffic Inflow
1
0.5
6
5
4
3
2
1
1
2
3
4
5
6


Employment Demand
2
1.5
1
0.5
6
5
4
3
2
1
1
2
3
4
5
6

# Commuting Flows

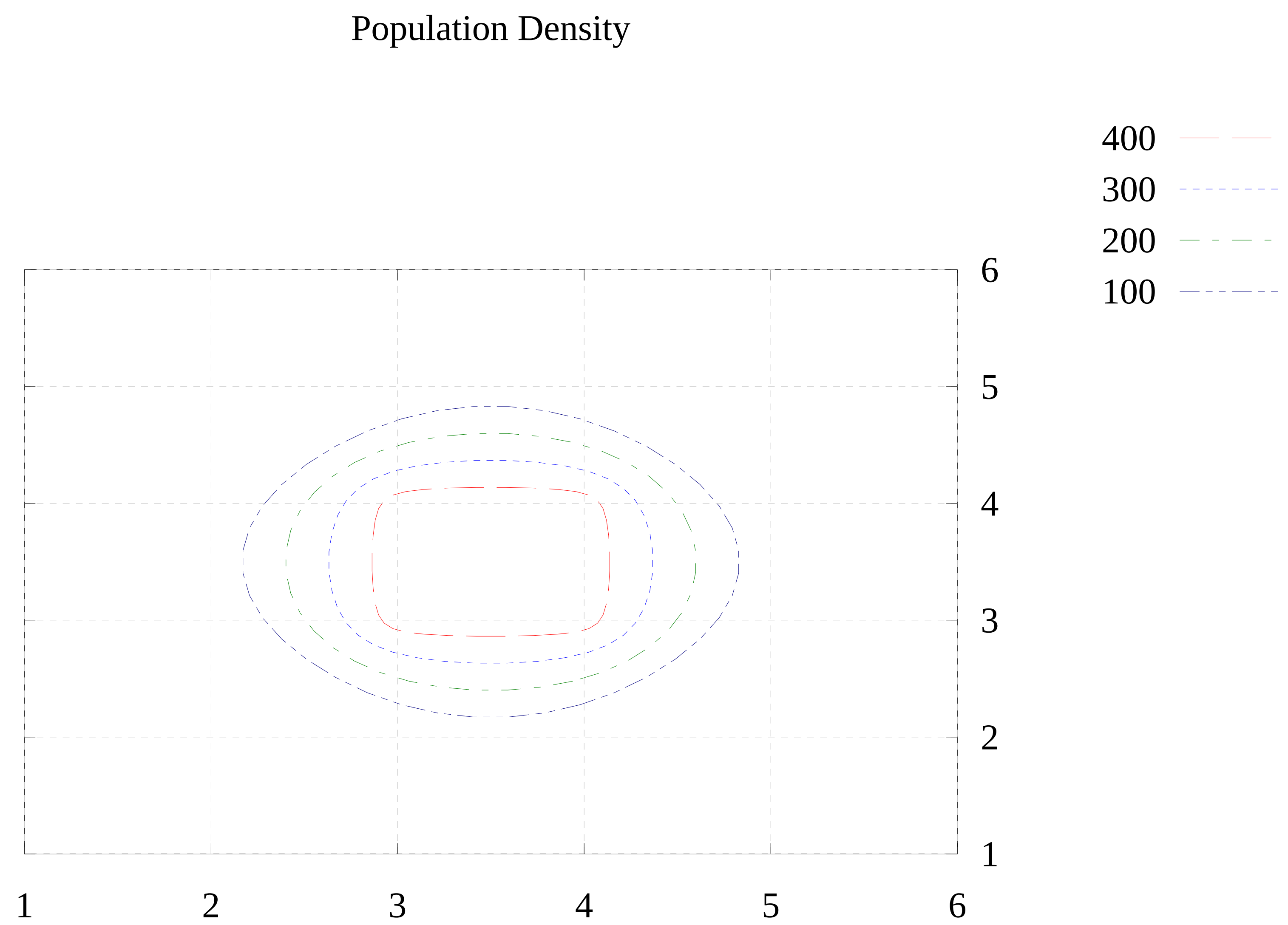


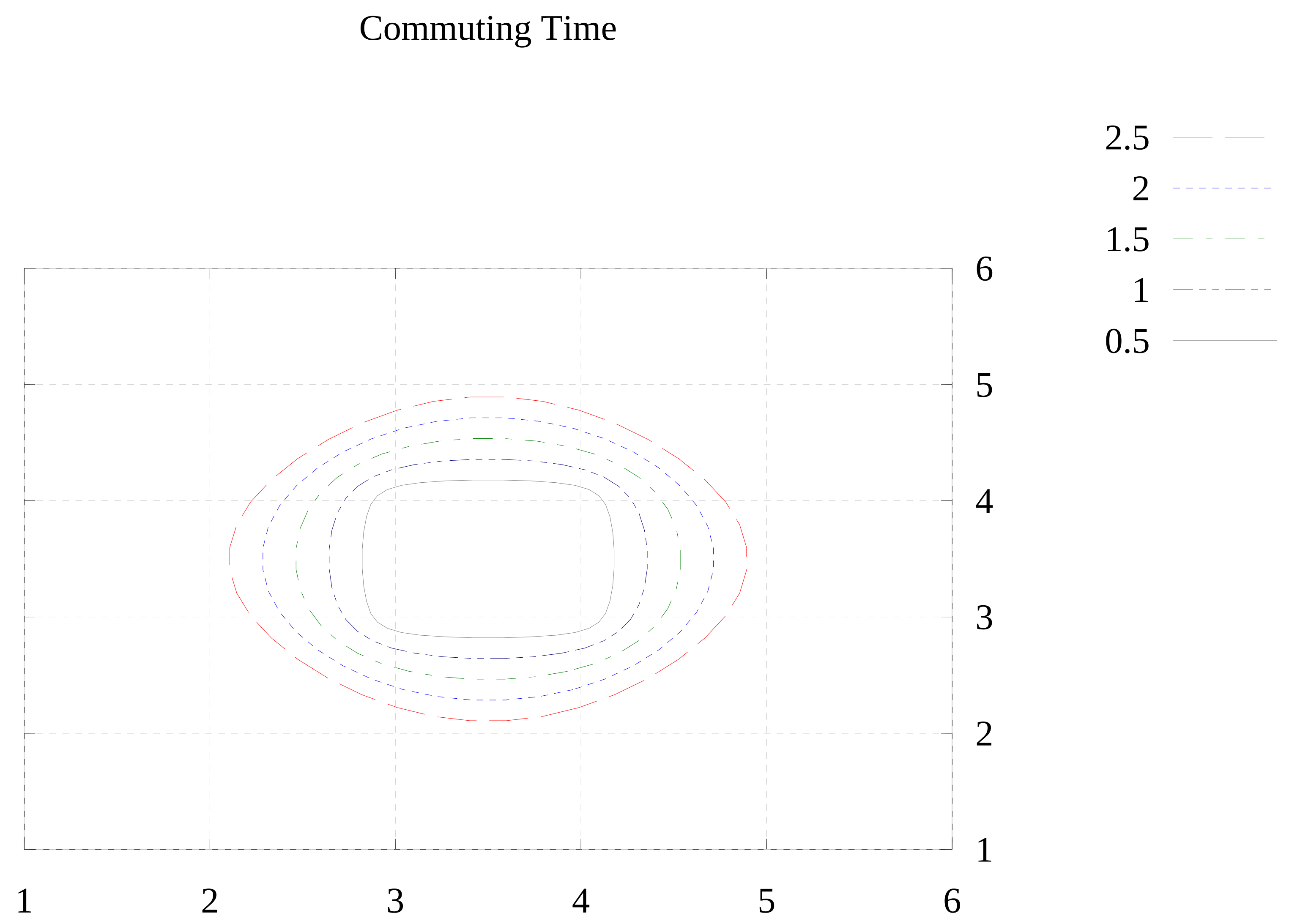
*Example 2: Jobs are located in the center*

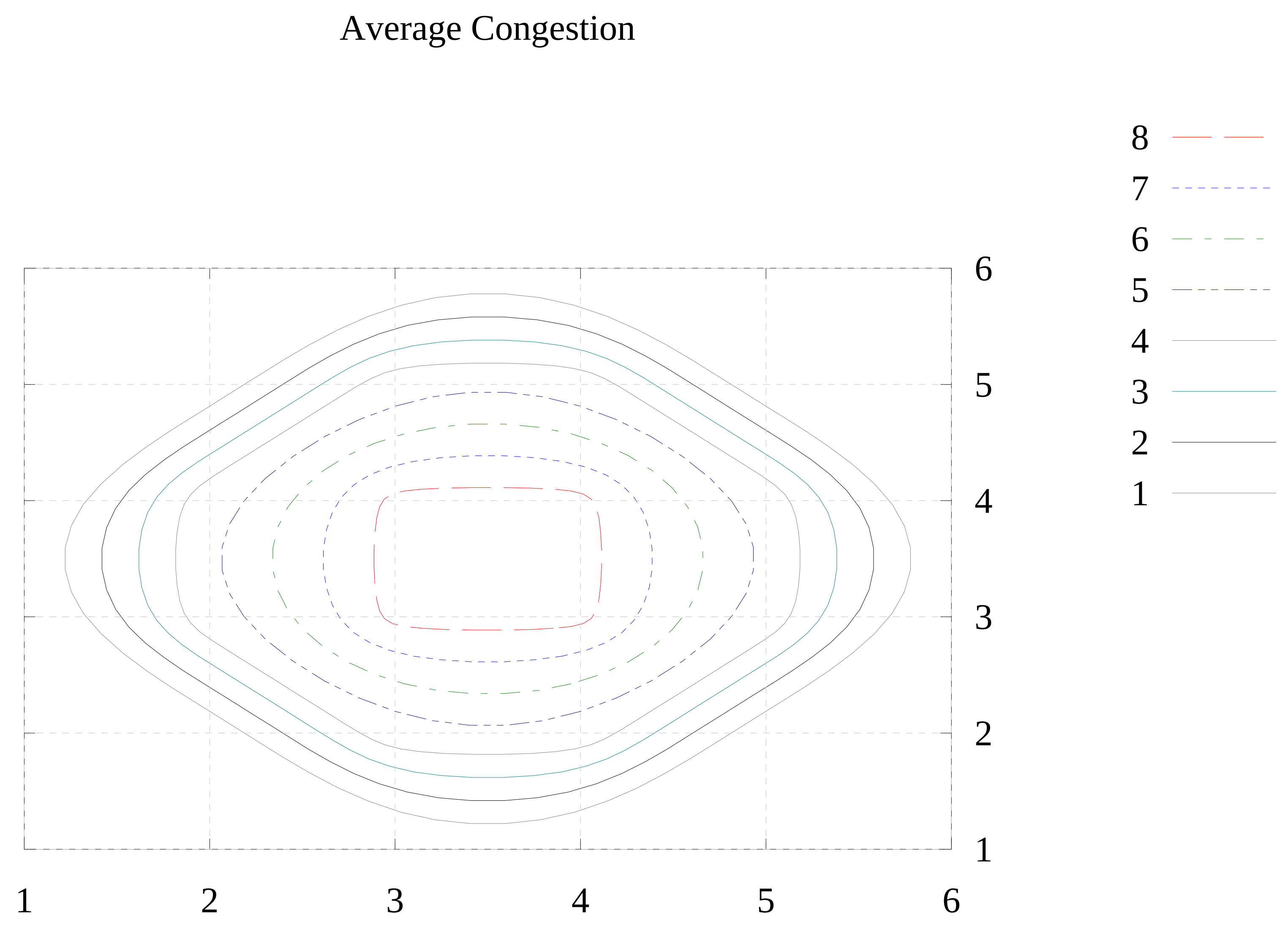


Traffic Inflow
3
2
1
6
5
4
3
2
1
1
2
3
4
5
6

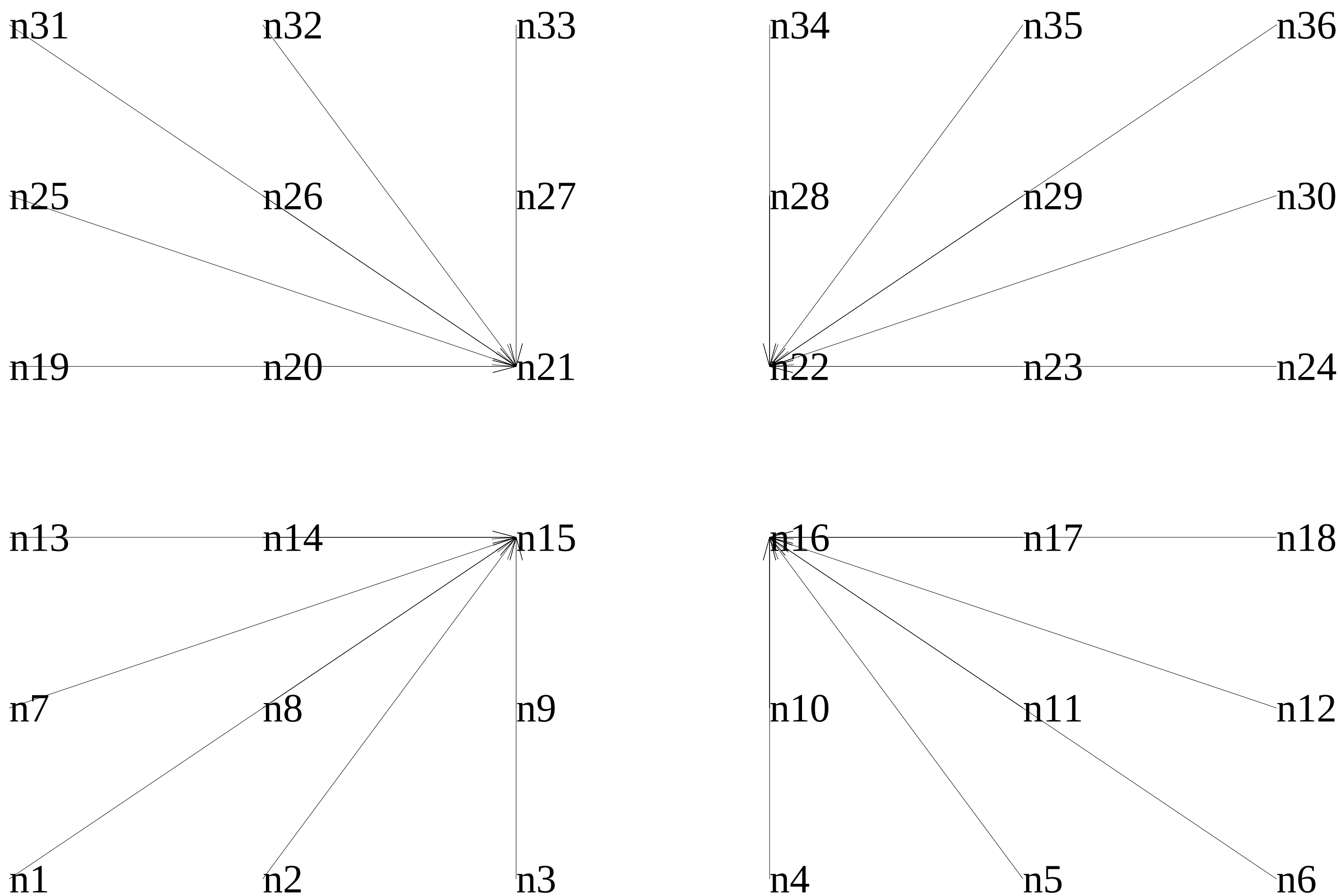

Housing Prices
5
4
3
2
1
6
5
4
3
2
1
1
2
3
4
5
6


Population Density
400
300
200
100
6
5
4
3
2
1
1
2
3
4
5
6

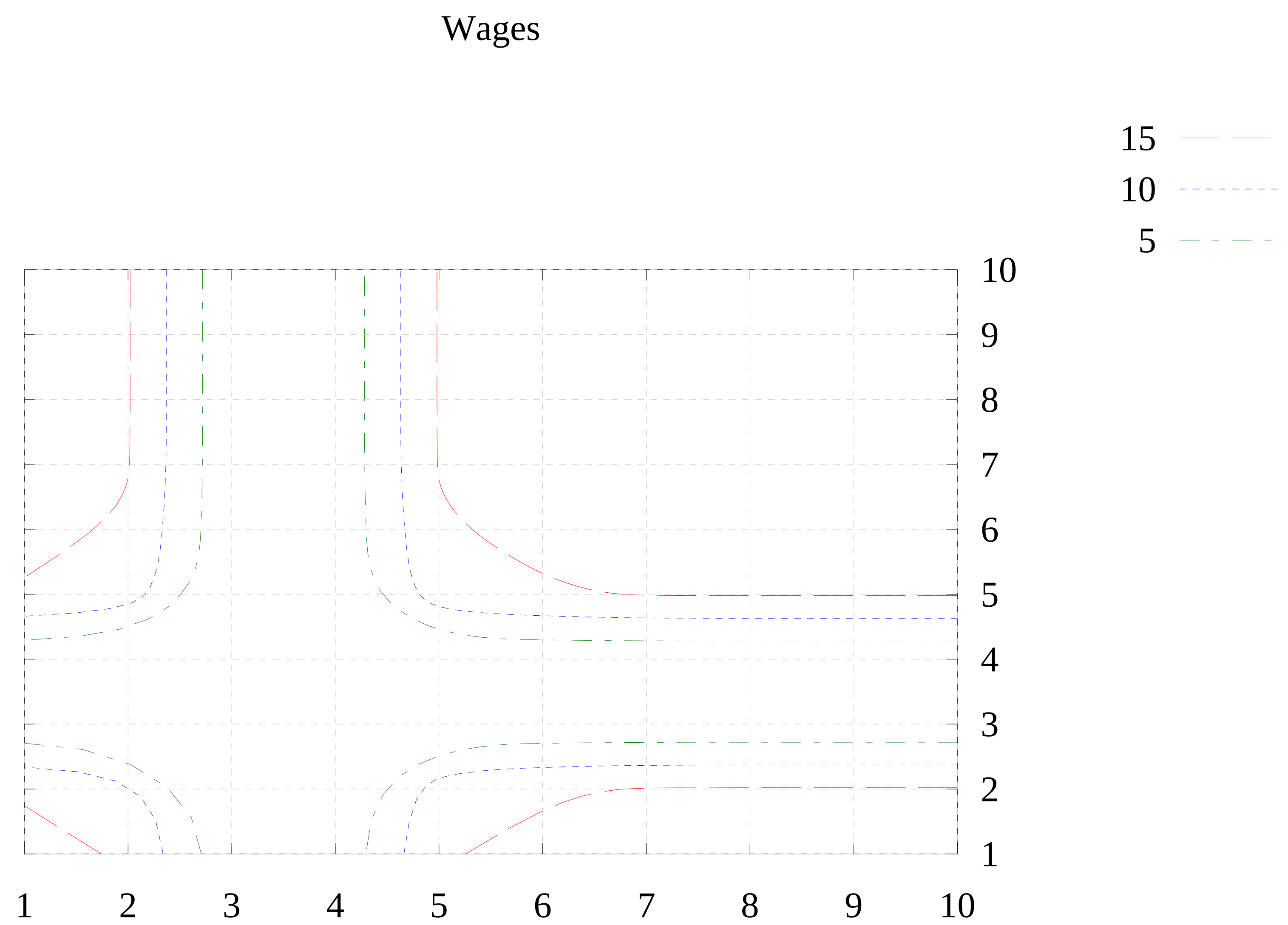

Commuting Time
2.5
2
1.5
1
0.5
6
5
4
3
2
1
1
2
3
4
5
6


Average Congestion
8
7
6
5
4
3
2
1
6
5
4
3
2
1
1
2
3
4
5
6

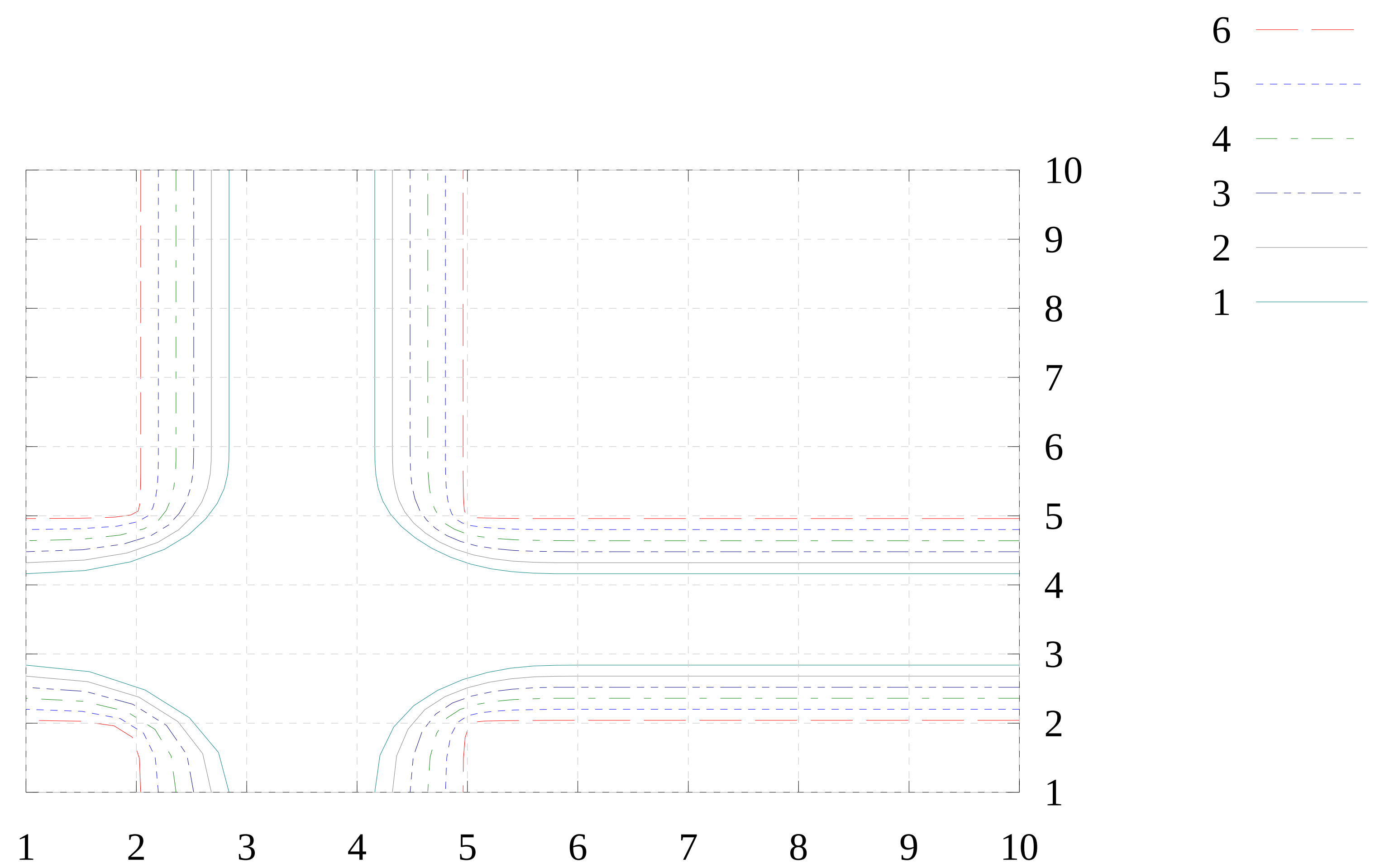
# Commuting Flows

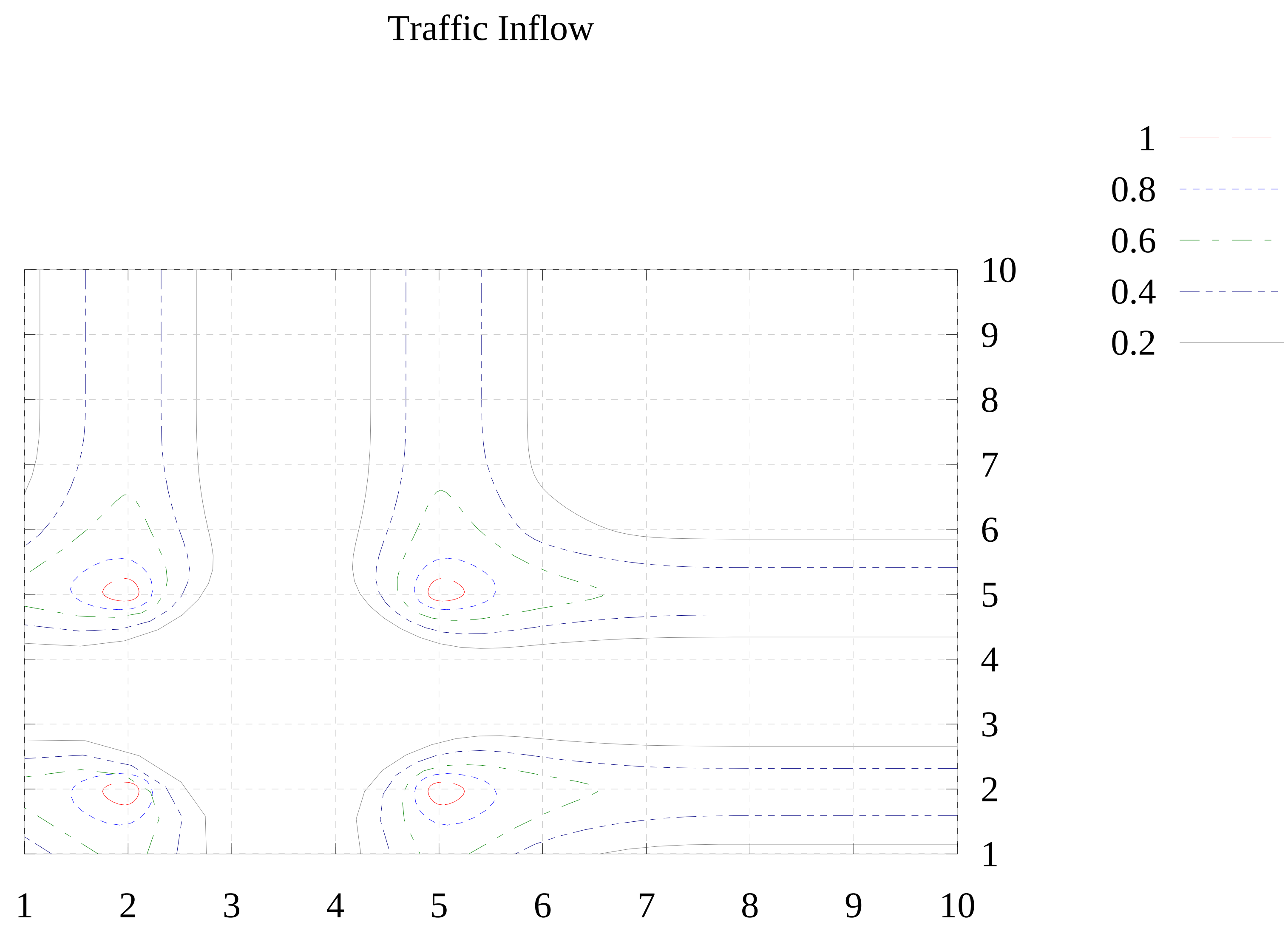


*Example 3: Jobs are located in the corners (assymetric)*

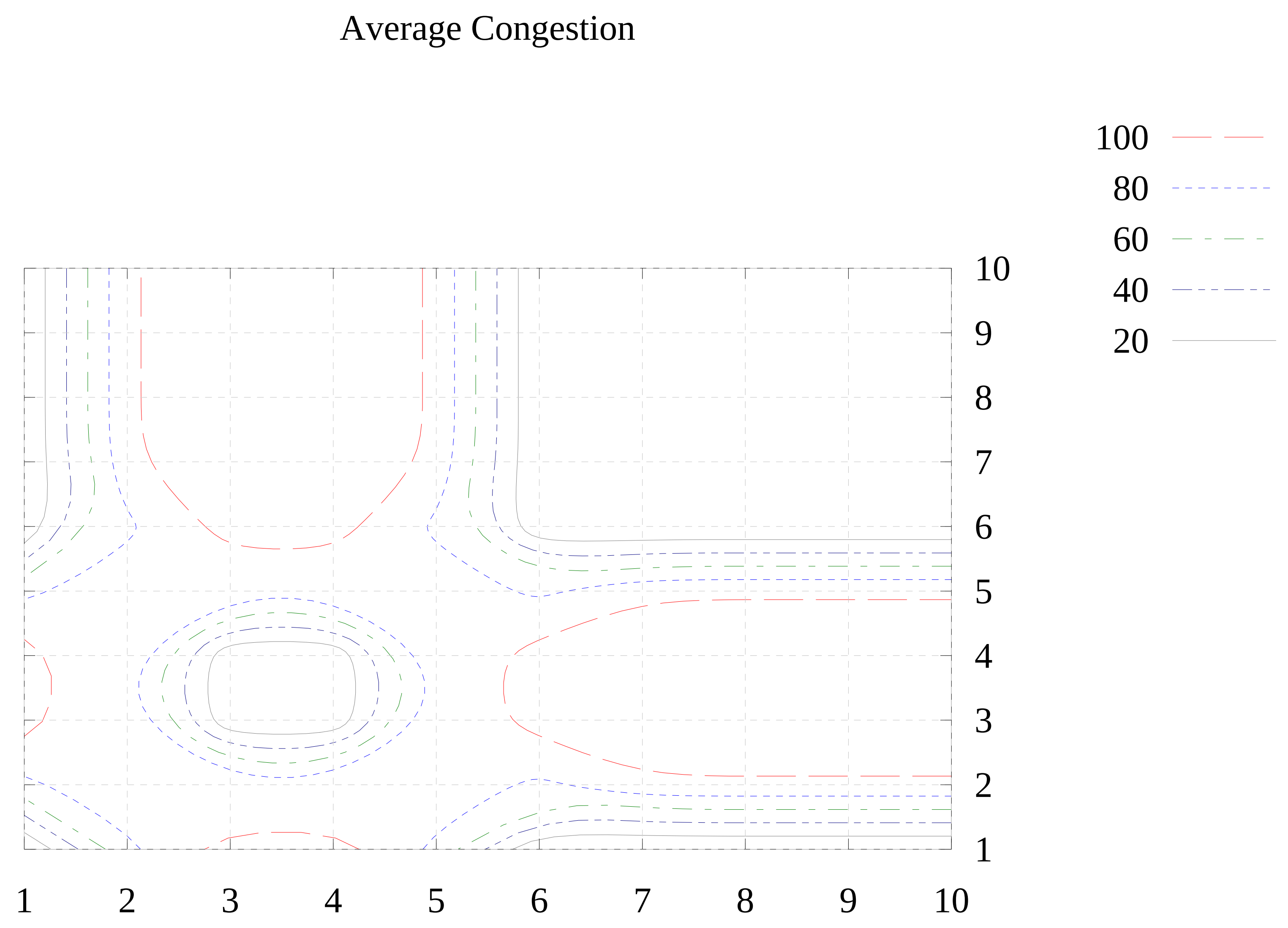

Wages
15
10
5
1 2 3 4 5 6 7 8 9 10
1 2 3 4 5 6 7 8 9 10

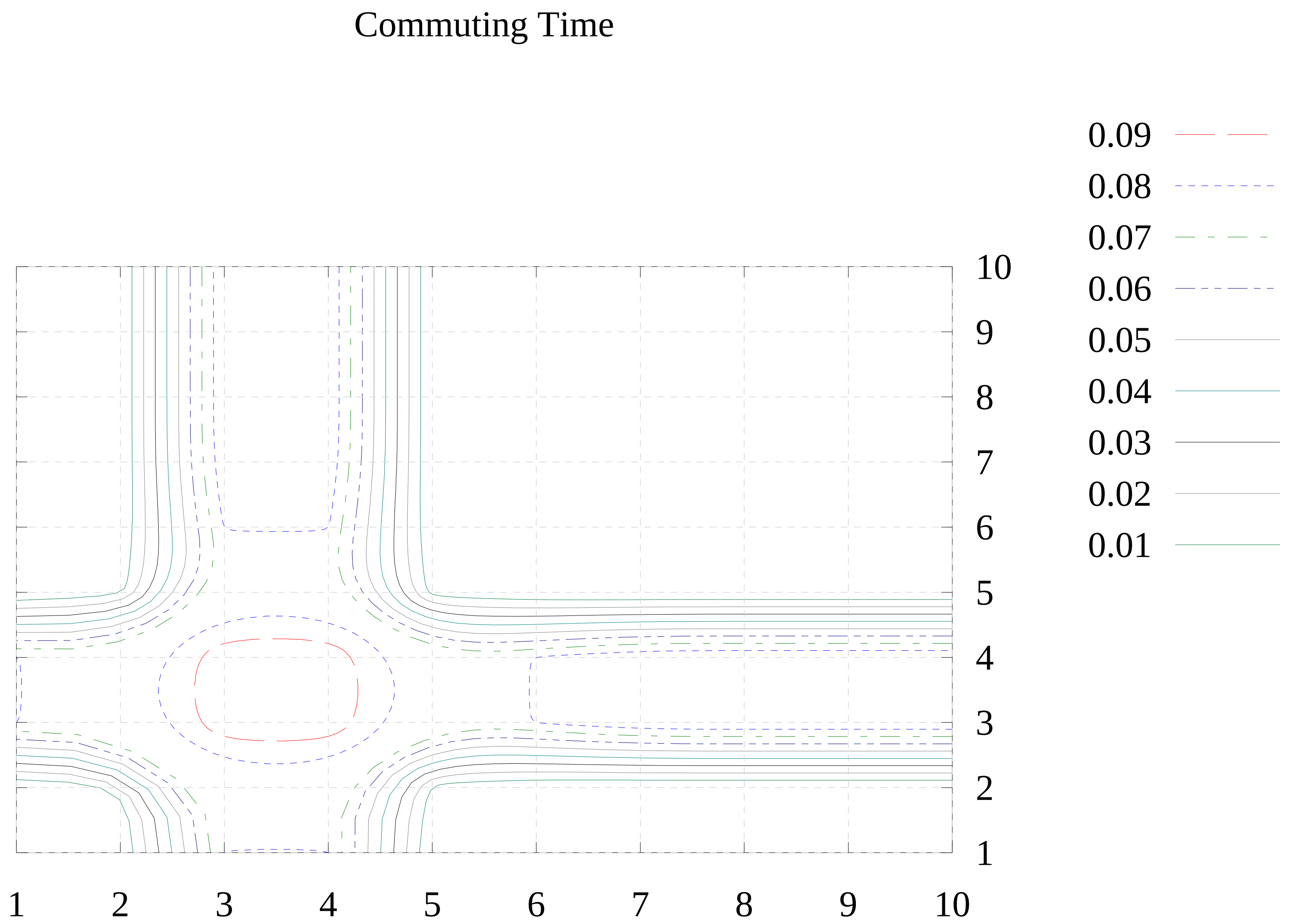
# Employment Demand



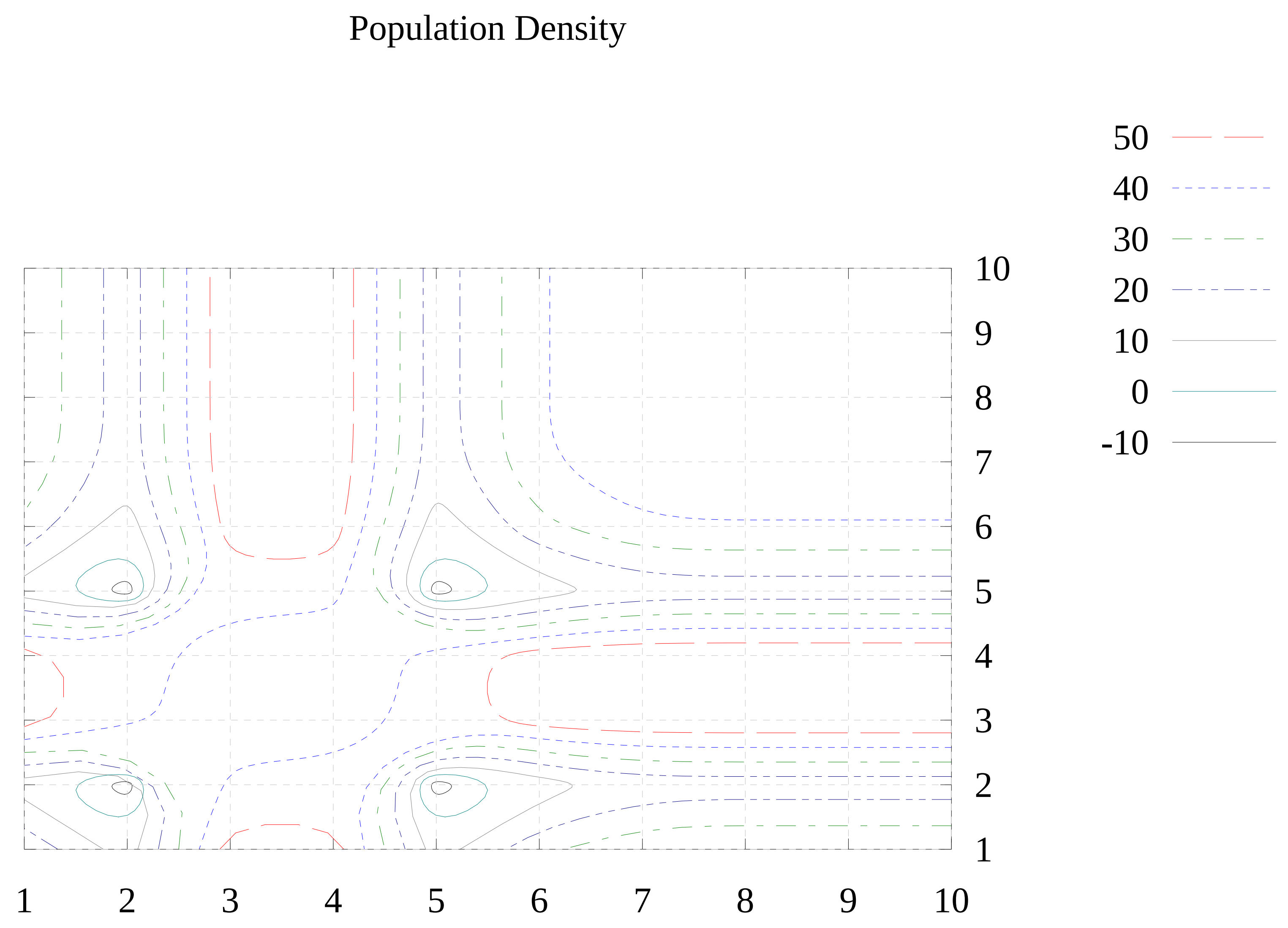


Traffic Inflow
1
0.8
0.6
0.4
0.2
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7
8
9
10

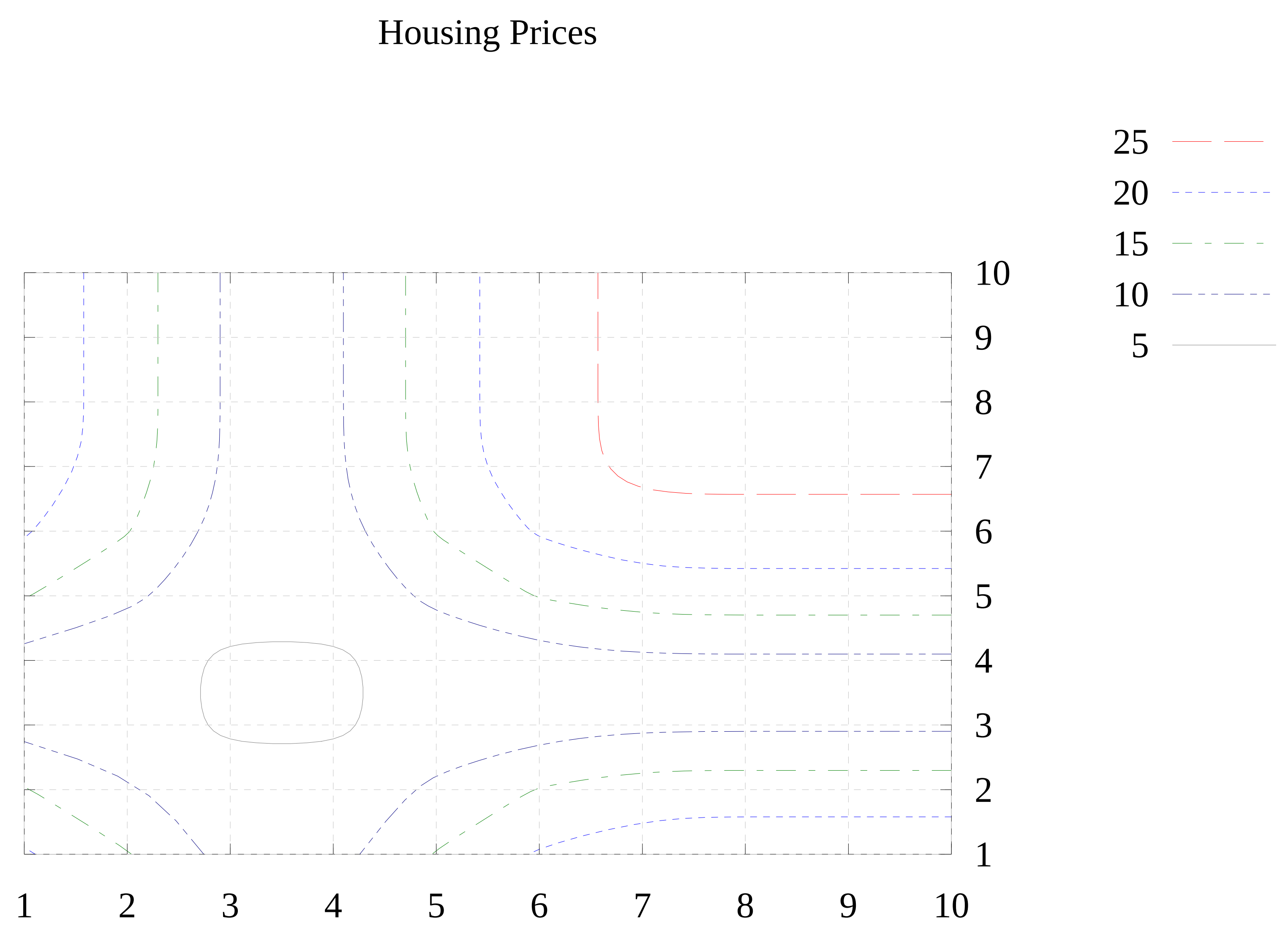

Average Congestion
100
80
60
40
20
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7
8
9
10



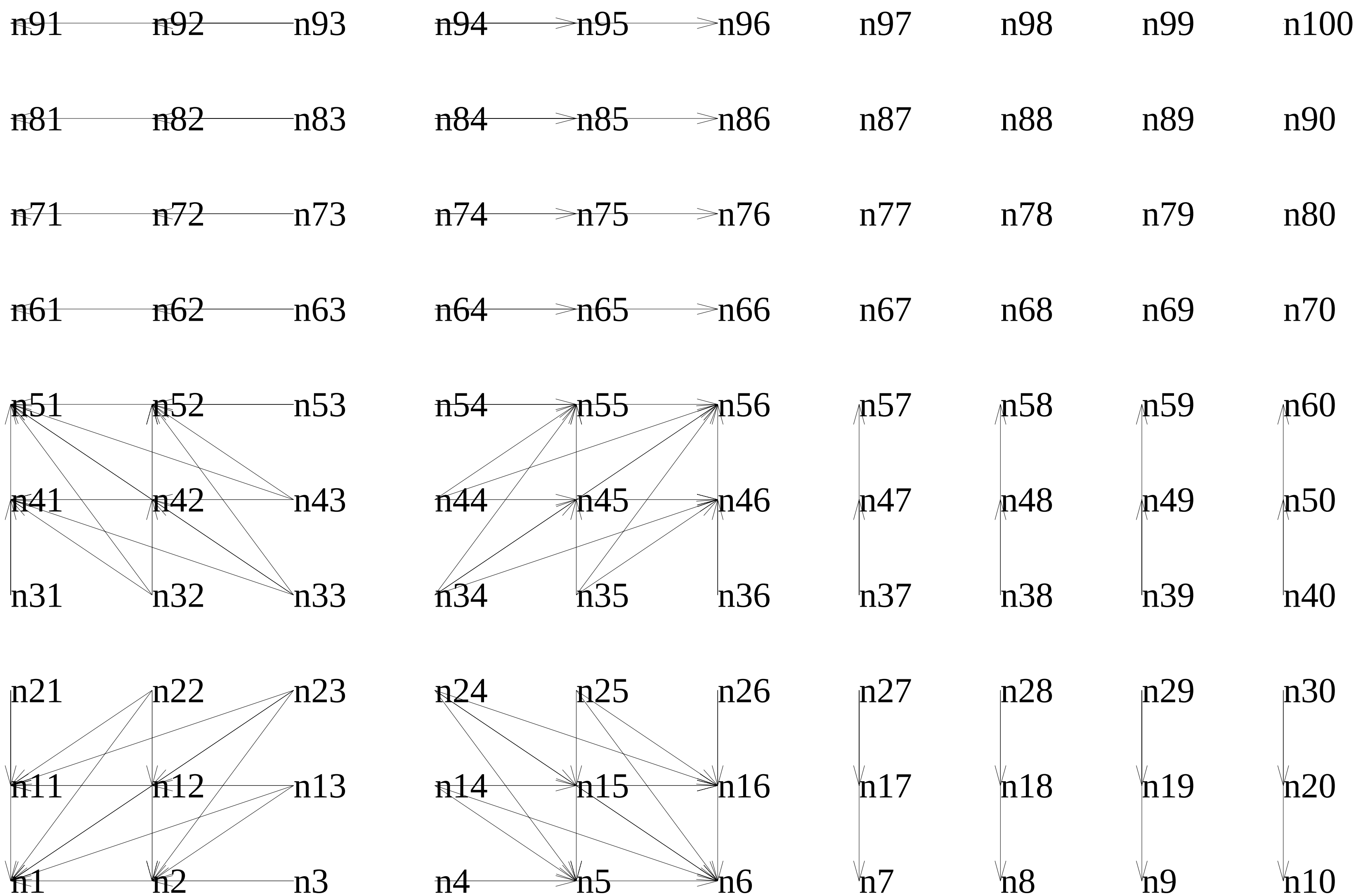
Commuting Time
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
10
9
8
7
6
5
4
3
2
1
1
2
3
4
5
6
7
8
9
10



Population Density
50
40
30
20
10
0
-10
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7
8
9
10


Housing Prices
25
20
15
10
5
1
2
3
4
5
6
7
8
9
10
1
2
3
4
5
6
7
8
9
10

Commuting Flows



*Example 4: Jobs are in the Northeast and House are in the Northwest*