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Introduction

- Motivation for cooperative game theoretic solutions

- Economies with many agents and small effective groups.

- The agenda:
  - Definition of a cooperative game and the core.
  - An example of a production economy.
  - Introduction to parameterized collections of games.
  - Equal treatment, monotonicity, and comparative statics.
  - An example of matching hospitals and interns.

– The relationship of parameterized collections of games to ‘pregames’ satisfying small group effectiveness.

GAMES

(N, v) – a *game,* where N is a finite set and v, called the *characteristic function,* maps subsets of N to $\mathbb{R}_+$ with $v(\phi) = 0$.

Nonempty subsets of N are called *groups*.

A *payoff vector* $x \in \mathbb{R}^N$ for (N, v) is *feasible* if

$$x(N) \stackrel{\text{def}}{=} \sum_{i \in N} x^i \leq \sum v(S^k)$$

for some partition $\{S^1, ..., S^K\}$ of N.

Given $\varepsilon \geq 0$, the payoff vector x is in the ε-*core* of the game if it is feasible and if, for all groups $S \subset N$,

$$x(S) \geq v(S) - \varepsilon |S|.$$

An example of a production economy

$b^t \in \mathbb{Z}^m$ – the resources of a player of type t, $t = 1, ..., T$.

$n = (n_1, ..., n_T) \in \mathbb{Z}_+^T - -$ (a *profile*) listing a number of players of each type. Each player of type t owns the resouces b^t.

$\sum_t s_t b^t := e(s)$ – the resources owned by a group of players with profile s. In the special case where each player's endowment consists of himself, his labor or simply a unit of his type it will hold that $e(s) = s$.

Assume that all coalitions have access to the same *technology* Λ for production of a good which sells at the price of \$1.00 per unit.

Given

$$n = (n_1, ..., n_T)$$

define N and a game (N, v) where

$$N = \{(t, q) : t = 1, ..., T \text{ and } q = 1, ..., f_t\},$$

$$v(S) = \Lambda(e(s)),$$

and s is defined by

$$s_t = |S \cap \{(t, q) : q = 1, ..., s_t\}|$$

Note: We could define another function directly on profiles of players, where the payoff function is given by

$$\Psi^O(s) := \Lambda(e(s)).$$

The pair (T, Ψ) is a *pregame.*

The *attribute core* – a set of prices for resources/ commodities/attributes $p^* \in \mathbb{R}^m_+$ such that

$$p^* \cdot e(n) = \Lambda(n)$$

and

$$p^* \cdot z \geq \Lambda(z) \text{ for all } z \leq e(n).$$

Example Two endowment vectors $b^1 = (1,0)$ and $b^2 = (0,2)$.

$$\Lambda(x,y) = \min\{x,y\}.$$

Given $s \in Z^2_+$, $\Psi(s) = \min\{s_1, s_2\}$

$N = \{1,2\}$. $e(\{1\}) = b^1$, $e(\{2\}) = b^2 \Rightarrow$

$$v(N) = 1$$

$$v(\{1\}) = v(\{2\}) = 0.$$

The core of the game is the set of points

$$\{(u_1,u_2) \in \mathbb{R}^2_+ : u_1 + u_2 = 1\}.$$

The attribute core prices solve the following minimization problem:

$$\text{minimize } p_1 + 2p_2$$
$$\text{subject to}$$
$$p_1 + p_2 \geq 1,$$

which has the unique solultion:

$$p_1^* = 1,\ p_2^* = 0$$

and gives equilibrium utilities of $\{u_1 = 1, u_2 = 0\}$.

Back to games

(N, v) is *essentially superadditive* if a group S can divide into subgroups and achieve the total payoff realizable by the subgroups.

The *superadditive cover* (N, v^s) of the game (N, v) where:

$$v^s(S) \stackrel{\text{def}}{=} \max \sum_k v(S^k)$$

and the maximum is taken over all partitions $\{S^k\}$ of S. Our results apply to both superadditive games and to superadditive cover games.

Given $\delta \geq 0$, two players i and j are δ-*substitutes* if for all groups $S \subset N$ with $i, j \notin S$, it holds that

$$|v(S \cup \{i\}) - v(S \cup \{j\})| \leq \delta.$$

When $\delta = 0$, players i and j are *exact substitutes*.

Parameterized collections of games

$\delta-$substitute partitions A partition $\{N[t]\}$ of N is a δ-*substitute partition* if all players in each subset are δ-substitutes for each other. The set $N[t]$ is interpreted as an *approximate type*.

(δ,T)- type games. Let $\delta \geq 0$ and let $T \in \mathbb{Z}_+$. A game (N, v) is a (δ, T)-*type game* if there exists a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N.

profiles. Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes.

A *profile* relative to $\{N[t]\}$ is a vector of $f \in Z_+^T$.

Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$.

Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β-effective B-bounded groups. Groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let $\beta \geq 0$, and let $B \in \mathbb{Z}_+$. A game (N, v) has *β-effective B-bounded groups* if for every $S \subset N$ there is a partition $\left\{S^k\right\}$ of S into subgroups with $\left|S^k\right| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta \left|S\right|.$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups*.

parametrized collections of games $\Gamma((\delta, T), (\beta, B))$. Let T, $B \in \mathbb{Z}_+$ and let $\delta, \beta \geq 0$. Define

$$\Gamma((\delta, T), (\beta, B))$$

to be the collection of all (δ, T)-type games that have β-effective B-bounded groups.

Example A δ-substitute partition and β-effective B-bounded groups.

Suppose that players can be ranked in the $[0, 1]$ interval so that if $i, j \in N$ and $i \geq j$ then i has a higher rank. We consider three different games, all with the same player set and the same ranking.

Let (N, v) be a game where the payoff to any two players is the sum of their ranks. Suppose also that the payoff $v(S)$ to any other group S is zero. Then for any $\beta \geq 0$ and any $B \geq 2$, the game has β-effective B-bounded groups. Given $\delta \geq 0$, if the distance between the ranks of players i and j is less than δ, then i and j are δ-substitutes, both for the game (N, v) and for the superadditive cover game (N, v^s).

Consider another game (N, v') but where the payoff $v'(\{i\})$ to player i is equal to his rank and the payoff to any other coalition is the given by the superadditive cover of v'. Here for any $\beta \geq 0$ and any $B \geq 1$, B-bounded groups are effective and if the distance between the ranks of i and j less than δ, then i and j are δ-substitutes.

Or, let the payoff to any group consisting of two players be the square of the sum of the ranks of the members of the group. If the distance between the ranks of players i and j is less than δ then i and j are $\delta^2 + 4\delta$ substitutes.

Equal treatment ε-core

the core and ε-cores. Take $\varepsilon \geq 0$. A payoff vector x is in the ε-*core* of (N, v) if and only if it is *feasible*, that is,

$$\sum_{a \in N} x_a \leq v(N)$$

and

$$\sum_{a \in S} x_a \geq v(S) - \varepsilon |S|$$

for all $S \subset N$.

the equal treatment ε-core. Given ε, $\delta \geq 0$, the *equal treatment ε-core* of (N, v) relative to a δ-substitute partition $\{N[t]\}$ is the set of payoff vectors x in the ε-core such that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

For our comparative statics and monotonicity results, we restrict to payoffs in equal treatment ε-cores.

Lemma. Let (N, v) be a not-necessarily superadditive game and let (N, v^s) be its superadditive cover. Let $\varepsilon \geq 0$ be given. Then if x is a payoff vector in the ε-core of (N, v), then x is in the ε-core of (N, v^s).

Proposition. Let $(N, v) \in \Gamma((\delta, T), (0, B))$. Let $z \in \mathbb{R}_+^N$ be in the core of (N, v). Suppose that there are more than B δ-substitutes for each player in the game. Then if i,$j \in N$ and i and j are δ-substitutes, it holds that

$$\left| z_i - z_j \right| \leq 2\delta.$$

Laws of scarcity

For $x, y \in \mathbf{R}^T$, $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1, 2$, let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Proof: Since (N, v) has β-effective B-bounded groups, there exists a partition $\left\{G^{1\ell}\right\}$ of S^1, such that $\left|G^{1\ell}\right| \leq B$ for any ℓ and

$$\sum_{\ell} v(G^{1\ell}) \geq v(S^1) - \beta \left\| f^1 \right\|.$$

Denote the profiles of $G^{1\ell}$ by g^{ℓ}. Observe that $\sum_{\ell} g^{\ell} = f^1$.

Since $f_t^2 \geq B$ for each t, it holds that $g^{\ell} \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^{ℓ}. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^{ℓ}, it holds that

$$\left| v(G^{1\ell}) - v(G^{2\ell}) \right| \leq \delta \left\| g^{\ell} \right\|.$$

Since x^2 represents a payoff vector in the equal treatment ε-core of (S^2, v) and $G^{2\ell} \subset S^2$ has profile g^{ℓ}, the total payoff $x^2 \cdot g^{\ell}$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \left\| g^{\ell} \right\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^{ℓ}, it holds that

$$\begin{aligned} x^2 \cdot g^{\ell} &\geq v(G^{2\ell}) - \varepsilon \left\| g^{\ell} \right\| \\ &\geq v(G^{1\ell}) - (\varepsilon + \delta) \left\| g^{\ell} \right\|. \end{aligned}$$

Adding these inequalities we have

$$x^2 \cdot f^1 \geq \sum_{\ell} v(G^{1\ell}) - (\varepsilon + \delta) \left\| f^1 \right\|.$$

It then follows that

$$x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Since x^1 represents a payoff vector in the equal treatment ε-core of (S^1, v), $x^1 \cdot f^1$ is feasible for (S^1, v), that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

■

Approximate cyclic monotonicity

We derive an exact bound on the amount by which an approximate core payoff vector for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on:

δ, the extent to which players within each of T types may differ from being exact substitutes for each other;

β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and

ε, a measure of the extent to which the ε-core differs from the core.

Proposition 1. Let (N,v) be in $\Gamma((\delta,T),(\beta,B))$ and let $(S^1,v),..,(S^K,v)$ be subgames of (N,v).

Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k.

Assume that $f_t^k \geq B$ for each k and each t.

For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k,v). Then

$$(x^1-x^2)\cdot f^1+(x^2-x^3)\cdot f^2+..+(x^K-x^1)\cdot f^K$$

$$\leq (\varepsilon+\delta+\beta)\left\|f^1+f^2+..+f^K\right\|$$

and

$$(x^1-x^2)\cdot \frac{f^1}{\|f^1\|}+(x^2-x^3)\cdot \frac{f^2}{\|f^2\|}+..+(x^K-x^1)\cdot \frac{f^K}{\|f^K\|}$$

$$\leq K(\varepsilon+\delta+\beta).$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity both in terms of numbers of players of each type and percentages of players of each type.

Remark. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 \right\| .$$

This form of monotonicity is typically called simply *monotonicity* or *weak monotonicity*.

Note that weak monotonicity does not imply cyclic monotonicity.

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 \right\|$$

and

$$(x^1 - x^2) \cdot (\frac{f^1}{\left\| f^1 \right\|} - \frac{f^2}{\left\| f^2 \right\|}) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

Note that the bound of Proposition 1 and its Corollary holds for any partition of the player set into δ-substitutes.

Comparative statics

Let $e^j \in \mathbf{R}^T$ such that $e^j_l = 1$ for $l = j$ and 0 otherwise.

Proposition 2. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v).

Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k.

Assume that $f^k_t \geq B$ for each k and each t.

For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then the following holds:

(A) If $f^2 = f^1 + me^j$ for some positive integer m (i.e., the second game has more players of approximate type j but the same numbers of players of other types) then

$$(x_j^2 - x_j^1)$$

$$\leq (\varepsilon + \delta + \beta)\frac{\|f^1+f^2\|}{\|f^2-f^1\|}$$

$$= (\varepsilon + \delta + \beta)\frac{2\|f^2\|-m}{m}.$$

(B) If

$$\frac{f^2}{\left\|f^2\right\|} = (1-\mu)\frac{f^1}{\left\|f^1\right\|} + \mu e^j$$

for some $\mu \in (0,1)$ (i.e., the second game has proportionally more players of approximate type j but the same proportions between the numbers of players of other types) then

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta)\frac{2-\mu}{\mu}.$$

That is, approximately the equal treatment ε-core correspondence provides lower payoffs for players of a type that is more abundant.

Proof: (A): Applying Corollary we get $(x^2 - x^1) \cdot me^j \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 \right\|$. Since $\left\| f^2 \right\| = \left\| f^1 \right\| + m$, this inequality implies our first result.

(B): From Lemma 1 we have $(1-\mu)(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} \leq (1-\mu)(\varepsilon + \delta + \beta)$ and similarly $(x^2 - x^1) \cdot \frac{f^2}{\|f^2\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain $(x^2 - x^1) \cdot (\frac{f^2}{\|f^2\|} - (1-\mu)\frac{f^1}{\|f^1\|}) \leq (2-\mu)(\varepsilon + \delta + \beta)$. Thus we get that $(x^2 - x^1) \cdot \mu e^j \leq (2-\mu)(\varepsilon + \delta + \beta)$. This inequality implies our second result. ■

Obviously, again the bounds provided by Proposition 2 are independent of the specific partition of the player set into δ-substitutes.

Further Remarks.

2. Note that the bounds on the closeness of all our results are computable for a given game and depend only on the parameters describing the game.

3. For $\varepsilon = \beta = \delta = 0$ the bounds on the closeness of all our approximation results equals zero. Thus for games with finite number of player types and strictly effective small groups (e.g. for matching games with types) the equal treatment core satisfies cyclic monotonicity and when a type becomes more abundant, players of that type receive (weakly) lower payoffs.

4. The results stated all require that there be at least B players of each type in each game under consideration. With other notions of approximate cores, specifically, the ε-remainder core and the ε_1-remainder ε_2-core, which allow a small percentage of players to be ignored, it may only be required that there are many substitutes for most players in the game.

5. Results similar to those herein could be obtained for the strong ε-core. This approximate core notion requires that no group of agents can improve on a given payoff by ε in total, that is, given a game (N, v) and $\varepsilon \geq 0$, a payoff vector x is in the *strong ε-core* of (N, v) if and only if

$$\sum_{a \in N} x_a \leq v(N)$$

and $\sum_{a \in S} x_a \geq v(S) - \varepsilon$ for all $S \subset N$.

Matching hospitals and interns; an example

Consider the assignment of a set of interns $\mathcal{I} = \{1, .., i, .., I\}$ to hospitals.

The set of hospitals is $\mathcal{H} = \{1, .., h, .., H\}$.

$N = \mathcal{I} \bigcup \mathcal{H}$.

Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ.

Interns also have preferences over hospitals.

Assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$).

Assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's *quality*.

Assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. The rank assigned to any intern is between 0 and 1 and the rank assigned to any hospital is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0, 1)$ and similarly, all hospitals have qualities in the interval $[1, 2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0, 2]$ into $2n$ intervals, $[0, \frac{1}{n}), .., [\frac{j-1}{n}, \frac{j}{n}), .., [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that if there is a player with rank in the

jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payoff vector in the ε-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x_j^2 - x_j^1) \leq (\varepsilon + \frac{1}{n}) \frac{\left\| f^1 + f^2 \right\|}{\left\| f^2 - f^1 \right\|}.$$

This is not the most general application of our results – we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

The parameter values that we have used in this example were chosen for convenience and simplicity. In principle, these could be estimated and various questions addressed. For example, are payoffs to interns approximately competitive? Do non-market characteristics such as ethnic background or gender make significant differences to payoffs?

Note that our results imply a certain *continuity* of comparative statics results with respect to changes in the descriptors of the total player set. In particular, the results are independent of the exact partition of players into approximate types. Specifically, given a number T of approximate types and a measure of the required closeness of the approximation, subject to the condition that players of each type are approximate substitutes for each other, our results apply independently of exactly where the boundary lines between types are drawn.

Note that the bounds obtained are *exact bounds*, in terms of the parameters describing a game.

Attribute games

One interpretation of a game with side payments, common in the literature, is to regard the players in the game as commodities or inputs. We call this an *attribute game* and the equal treatment core is called the *attribute core*. Our results immediately apply to attribute games.

For a simple example, consider a glove game where each player is a RH glove or a LH glove and the payoff to a coalition consisting of n_1 RH glove players and n_2 LH glove players is $\Psi(n_1, n_2) := \min\{n_1, n_2\}$. Suppose that in total, there are f_1 RH gloves and f_2 left hand gloves. Our laws of scarcity apply equally well to this interpretation of a game. Note that this game is a member of the collection $\Gamma((0, 2), (0, 2))$.

If ownership of *bundles* of commodities is assigned to individual units (teams or divisions within a firm in the literature on subsidy-free pricing or endowments of individual consumers of commodities in the exchange economy interpretation), then another cooperative game is generated. In this game, essentially some players in the original game are "syndicated," glued together to become one player.

From the data given above, we can construct games where players may be endowed with bundles of gloves. By endowing players in this game with various numbers of RH gloves and LH gloves, we create another game with possibly several types of players. For specificity, suppose:

1. m_1 players of type 1 are endowed with two right hand gloves each;

2. m_2 players of type 2 are endowed with a RH glove and;

3. m_3 players of type 3 are endowed with a LH glove.

For consistency, it must hold that $2m_1 + m_2 = f_1$ and $m_3 = f_2$. From the data given, with the three possible endowments of gloves given by 1-3 above, we can determine a number of types T and a bound B so that the game constructed, say (M, w), is a member of the collection $\Gamma((0, T), (0, B))$. It is immediate that $T = 3$.

It is fairly obvious that $B = 3$ suffices; the largest coalitions that need form in realizing all gains to collective activities consist of one player of type 1 and two players of type 3.

Thus, we have that $(M, w) \in \Gamma((0, 3), (0, 3))$ and our comparative statics and monotonicity results apply to the games in $\Gamma((0, 3), (0, 3))$.

Small Group Effectiveness

Recall that pregame (T, Ψ) is a finite number of types T and a function Ψ from $\mathbb{Z}_+^T$ to $\mathbb{R}_+$.

A pregame (T, Ψ) satisfies *small group effectiveness*, (SGE), if for each positive real number $\beta > 0$ there is an integer $\eta_1(\beta)$ such that for each profile f, for some partition $\{f^k\}$ of f :

(a) $\|f^k\| \leq \eta_1(\beta)$ for each profile f^k in the partition,
(b) $\Psi(f) - \sum_k \Psi(f^k) < \beta\|f\|$.

Let (T, Ψ) satisfy small group effectiveness and let β and $\eta_1(\beta)$ satisfy the condition of the definition of SGE. Then it is immediate that any game generated by the pregame has β-effective $\eta_1(\beta)$-bounded groups. Since T is a compact metric space it holds that given $\delta > 0$ we can partition T into a finite number T of subsets so that all players with attributes in each subset are δ-substitutes. Thus, all games derived from (T, Ψ) are in the collection $\Gamma((\delta, T), (\beta, B))$.

When games are required to have many substitutes for each player, small group effectiveness is equivalent to per capita boundedness. A pregame (T, Ψ) satisfies *per capita boundedness* if there is a constant A such that

$$\frac{\Psi(f)}{\|f\|} \leq A \text{ for all profiles } f \in P(T).$$

The following result holds more generally but is proven for the case where T is a finite set.

Theorem. *With "thickness," SGE=PCB.*

(1) Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.

(2) Let (T, Ψ) be a pregame satisfying PCB. Then given any positive real number ρ, construct a new pregame (T, Ψ_ρ) where the domain of Ψ_ρ is restricted to profiles f where, for each $t = 1, \cdots, T$, either $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$ (thickness).

Then (T, Ψ_ρ) satisfies SGE on its domain.

SGE requires that almost all feasible gains to collective activities can be achieved by groups bounded in absolute size. A pregame (T, Ψ) satisfies *small group effectiveness* *for improvement* if for each positive real number $\epsilon > 0$ there is an integer $\eta_2(\epsilon)$ with the following property:

For any profile f and any payoff function $x : \sigma(f) \to R_+$
if $x(f) + \epsilon \|f\| < \Psi(f)$ then there is a subprofile g of f such th

$$\|g\| \leq \eta_2(\epsilon) \text{ and } x(g) + \tfrac{\epsilon}{2} \|g\| < \Psi(g).$$

These two concepts are equivalent.

The pregame framework may also hide what makes the results work – the facts that there are many close substitutes for most players and that groups bounded is size can nearly exhaust gains to collective activities. In addition, since the pregame framework specifies payoffs for all groups, no matter how large, in general it is difficult, if not impossible to estimate the pregame function Ψ. In contrast, within the framework of parameterized collections, there are only four parameters to be estimated – δ, T, β, and B. The notion of β-effective B-bounded groups makes explicit how close coalitions bounded in size by B are to being able to realize all gains to collective activities for a given game.

Motivation for cooperative game theoretic solutions.

Standard models of price taking equilibrium are highly stylized. For example, as mentioned already several times during the last two days, typically convexity (or concavity) is required to ensure existence of solutions. But many interesting economic problems are inherently nonconvex, for example:

1. Production. (Think of the standard intermediate micro economics production function example.)

2. Coalition production, where all groups of economic actors may have a different production set available to them, depending on the characteristics of the members of the group.

3. Economies with indivisibilities and nonmonotonicities.

4. Economies with clubs or local public goods subject to congestion – there may be an optimal club or jurisdiction size smaller than the entire economy.

One approach to these sorts of problems is to assume that there is a continuum of agents, study properties of an equilibrium in this economy, and then use this equilibrium to estimate properties of equilibrium of large finite economies. In this presentation, I'd like to introduce you to another approach, currently in development. This will be done in the context of economies with quasi-linear utilities (or, in game theoretic terms, cooperative games with side payments). More general models exist in the literature (for example, papers of mine with John Conley and Alexander Kovalenkov in *Journal of Economic Theory* 2001 and 2003).

In effect, a few apparently very mild assumptions drive many of the features of large economies. Game theoretic solution concepts such as the core (in various forms) can represent the competitive equilibrium and be applied to diverse situations.

What may be lost is uniqueness of solutions. Is this a problem? If it is a problem in reality, then highly stylized models with unique equilibrium outcomes are not a solution.

Games with player types

Let (N, v) be a game and suppose that, for some vector $n \in \mathbb{Z}_+^T$,

$$N = \{(t, q) : t = 1, ..., T,\ q = 1, ..., n_t\}$$

and, for each t, all players of type t are substitutes for each other. Then (N, v) is a game with types and n is *the profile of* N.

Let $S \subset N$. Define the *profile of* S as $s = (s_1, ..., s_T) \in \mathbb{Z}_+^T$ where, for each t,

$$s_t = |\{(t, q) : q = 1, ..., n_t\} \cap S| .$$

Define a function Ψ by

$$\Psi(s) = v(S)$$

for all groups $S \subset N$ where S has profile s.

We can describe the ε-core of the game as a solution to the following linear programming problem.

minimize $p \cdot n$

subject to

$p \cdot s \geq \Psi(s)$ for all profiles $s \leq n$.

Let p^* satisfy

$$p^* \cdot n = \min p \cdot n$$

subject to the constraints. Then p^* represents an equal treatment payoff (one that treats all players of the same type equally) if and only if

$$p^* \cdot n = \Psi(n).$$

It holds that, given a pregame (T, Ψ), and $\varepsilon > 0$, all derived games with sufficiently many players have nonempty ε-cores. (Wooders 1983,...., Kovalenkov and Wooders – recent papers in *MOR, JET,* and *GEB*).

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Economic Theory

Laws of scarcity for a finite game – exact bounds on estimations*

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Summary. A "law of scarcity" is that scarceness is rewarded. We demonstrate laws of scarcity for cores and approximate cores of games. Furthermore, we show that equal treatment core payoff vectors satisfy a condition of cyclic monotonicity. Our results are developed for parameterized collections of games and exact bounds on the maximum possible deviation of approximate core payoff vectors from satisfying a law of scarcity are stated in terms of the parameters describing the games. We note that the parameters can, in principle, be estimated.

Keywords and Phrases: Monotonicity, Cooperative games, Clubs, Games with side payments (TU games), Cyclic monotonicity, Law of scarcity, Law of demand, Approximate cores, Effective small groups, Parameterized collections of games.

JEL Classification: C71, C78, D41.

1 Laws of scarcity, parameterized collections of games and equal treatment cores

This paper treats cooperative games with many players and provides some characterization results for approximate cores, outcomes that are stable against coalition

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This article is dedicated to Marcel (Ket) Richter, a distinguished researcher and a wonderful teacher and mentor to his students. We are delighted to contribute our paper to this special issue of *Economic Theory* in his honor.
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formation. An advantage of the framework of cooperative games over detailed models of economies is that models of games can accommodate the entire spectrum from games derived from economies with only private goods to games derived from economies with pure public goods. Thus, it is of interest to determine conditions on games ensuring that they are ‘market-like’ – that they satisfy analogues of well known properties of competitive economies. Important papers in this direction include Shubik [21], which introduced the study of large games as models of large private-goods economies, Shapley and Shubik [20], which demonstrated an equivalence between markets and totally balanced games and Wooders [26,27] demonstrating that games with many players are market games. Further motivation for the framework of cooperative games comes from Buchanan [2], who stressed the need for a general theory, including as extreme cases both purely private and purely public goods economies and the need for “a theory of clubs, a theory of cooperative membership.”

The current paper employs the framework of parameterized collections of games and obtains Laws of Scarcity, analogues of the celebrated Laws of Demand and of Supply of general equilibrium theory. Roughly, the Law of Demand states that prices and quantities demanded change in the opposite directions while, with inputs signed negatively, the Law of Supply states that quantities demanded as inputs and produced as outputs change in the same direction as price changes.[1] In the framework of a cooperative game, supply and demand are not distinct concepts. Thus, following [26] we refer to our results for games as Laws of Scarcity. Roughly, our results state that, if almost all gains to collective activities can be realized by groups of players bounded in size, then numbers of players who are similar to each other and core payoffs respond in opposite directions. If player types are thought of as commodity types while payoffs to players are thought of as prices for commodities, our Laws of Scarcity are closely related to comparative statics results for general equilibrium models with quasi-linear utilities. As we discuss in a section relating our paper to the literature, our results extend the literature in several directions.

As in our prior papers on parameterized collections of games,[2] a game is described by certain parameters: (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. An equal treatment payoff vector is defined to be a payoff vector that assigns the same payoff to all players of the same approximate type. We show that equal treatment ε-cores satisfy the property that numbers of players who are similar to each other and equal treatment ε-core payoffs respond in nearly opposite directions; specifically, we establish an exact upper bound on the extent to which equal treatment ε-core payoffs may respond in the same direction and this bound will, under some conditions, be small. We actually demonstrate a stronger result – equal treatment ε-core vectors and vectors

[1] The Law of Demand therefore rules out “Giffen goods” or treats compensated demands; see Mas-Colell, Whinston and Green [12], Sections 2.F and 4.C. This volume also provides a very clear exposition and further references.

[2] [6–8] and [28].

of numbers of players of each approximate type satisfy cyclic monotonicity.[3] In addition to cyclic monotonicity, we demonstrate a closely related comparative statics result: When the relative size of a group of players who are all similar to each other increases, then equal treatment ε-core payoffs to members of that group will not significantly increase and may decrease.

The conditions required on a game to obtain our results are that (i) each player has many close substitutes (a thickness condition) and (ii) almost all gains to collective activities can be realized by groups of players bounded in size (small group effectiveness – SGE). The first condition is frequently employed in economic theory. The second condition may appear to be restrictive, but in fact, if there are sufficiently many players of each type, then per capita boundedness (PCB) – finiteness of the supremum of average payoff – and SGE are equivalent.[4] Our results yield explicit bounds, in terms of the parameters describing the games, on the maximal deviation of equal treatment ε-core payoffs from satisfying exact monotonicity. Moreover, our framework allows some latitude in the exact specification of approximate types. These two considerations suggest that in principle our results can be well applied to estimate the effects on equal treatment ε-core payoffs of changes in the composition of the total player set. Note that all the bounds we obtain are exact, and depend on the parameters describing the games and on the ε of the ε-core.

For our results characterizing ε-cores of games to be interesting, it is important that under some reasonably broad set of conditions, ε-cores of large games are nonempty. Since Shapley and Shubik [19] showing nonemptiness of approximate cores of exchange economies with many players and quasi-linear utilities and Wooders [23, 24], showing nonemptiness of approximate cores of game with many players with and without side payments, there has been a number of further results. For parameterized collections of games, such results are demonstrated in [6–8] and [28]. The interest of our monotonicity results is further enhanced by results showing that approximate cores have the equal treatment property; in this regard, note that [26] shows that approximate cores of large games treat most similar players nearly equally. In research in progress, similar equal treatment results are demonstrated for parameterized collections of games (see [10] for a first result).

In the next section we define parameterized collections of games. In Section 3, the results are presented. Section 4 consists of an example, applying our results to a matching model with hospitals and interns. Section 5 further relates the current paper to the literature and concludes the paper. In the Appendix we prove that the bounds cannot be tightened.

2 Cooperative games

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *characteristic function*, from subsets of N to the non-negative real

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[3] Cyclic monotonicity relates to monotonicity in the same way as the Strong Axiom of Revealed Preference relates to the Weak Axiom of Revealed Preference (see, for example, Richter [13,14]).

[4] This is shown for "pregames" in [27], Theorem 4. Per capita boundedness and small group effectiveness were introduced into the study of large games in Wooders [24, 25] respectively.

numbers with $v(\emptyset) = 0$. The pair (N, v) is a *game* (*with side payments* or a *TU game*). Non-empty subsets of N are called *coalitions* or *groups*. A game (N, v) is *superadditive* if $v(S) \geq \sum_k v(S^k)$ for all groups $S \subset N$ and for all partitions $\{S^k\}$ of S. For the current paper we restrict our attention to superadditive games. This is slightly more restrictive than required, but simplifies notation and shortens the proof.

2.1 Parameterized collections of games

$\delta-$substitute partitions. In our approach we approximate games with many players, all of whom may be distinct, by games with player types.

Let (N, v) be a game and let $\delta \geq 0$ be a non-negative real number. Informally, a δ-substitute partition is a partition of the player set N into subsets with the property that any two players in the same subset are "within δ" of being substitutes for each other. That is, if players in a coalition are replaced by δ-substitutes, the payoff to that coalition changes by no more than δ per capita. Formally, given a partition $\{N[t] : t = 1, .., T\}$ of N, a permutation τ of N is *type consistent* if, for any $i \in N$, $\tau(i)$ belongs to the same element of the partition $\{N[t]\}$ as i. A δ-*substitute partition* of N is a partition $\{N[t] : t = 1, .., T\}$ of N with the property that, for any type-consistent permutation τ and any coalition S,

$$|v(S) - v(\tau(S))| \leq \delta |S| .$$

Note that in general a δ-substitute partition of N is not uniquely determined. Moreover, two games, say (N, v) and (N, v'), may have the same partitions into δ-substitutes but have no other relationship to each other (in contrast to games derived from a pregame).

(δ,T)- *type games*. The notion of a (δ,T)-type game is an extension of the notion of a game with a finite number of types to a game with approximate types.

Let δ be a non-negative real number and let T be a positive integer. A game (N, v) is a (δ, T)-*type game* if there exists a T-member δ-substitute partition $\{N[t] : t = 1, .., T\}$ of N. The set $N[t]$ is interpreted as an *approximate type*. Players in the same element of a δ-substitute partition are δ-*substitutes*. When $\delta = 0$, they are *exact substitutes*.

profiles. Profiles of player sets are defined relative to partitions of player sets into approximate types.

Let $\delta \geq 0$ be a non-negative real number, let (N, v) be a game and let $\{N[t] : t = 1, .., T\}$ be a partition of N into δ-substitutes. A *profile* relative to $\{N[t]\}$ is a vector of non-negative integers $f \in Z_+^T$. Given $S \subset N$ the *profile of* S is a profile, say $s \in Z_+^T$, where $s_t = |S \cap N[t]|$. A profile describes a group of players in terms of the numbers of players of each approximate type in the group. Let $\|f\|$ denote the number of players in a group described by f, that is, $\|f\| = \sum f_t$.

β-effective B-bounded groups. The following notion formulates the idea of small group effectiveness, SGE, in the context of parameterized collections of games.

Informally, groups of players containing no more than B members are β-effective if, by restricting coalitions to having fewer than B members, the per capita loss is no more than β.

Let β be a given non-negative real number, and let B be a given integer. A game (N, v) has *β-effective B-bounded groups* if for every group $S \subset N$ there is a partition $\{S^k\}$ of S into subgroups with $|S^k| \leq B$ for each k and

$$v(S) - \sum_k v(S^k) \leq \beta |S| .$$

When $\beta = 0$, 0-effective B-bounded groups are called *strictly effective B-bounded groups*.

parametrized collections of games $\Gamma((\delta, T), (\beta, B))$. Let T and B be positive integers, let δ and β be non-negative real numbers. Define

$$\Gamma((\delta, T), (\beta, B))$$

to be the collection of all (δ, T)-type games that have β-effective B-bounded groups.

2.2 Equal treatment ε-core

the core and ε-cores. Let (N, v) be a game and let ε be a nonnegative real number. A payoff vector x is in the *ε-core* of (N, v) if and only if $\sum_{a \in N} x_a \leq v(N)$ and $\sum_{a \in S} x_a \geq v(S) - \varepsilon |S|$ for all $S \subset N$. When $\varepsilon = 0$, the ε-core is the *core*.

the equal treatment ε-core. Given nonnegative real numbers ε and δ, we will define the *equal treatment ε-core* of a game (N, v) relative to a δ-substitute partition $\{N[t]\}$ of the player set as the set of payoff vectors x in the ε-core with the property that for each t and all i and j in $N[t]$, it holds that $x_i = x_j$.

With the definition of the equal treatment ε-core in hand, we can next address monotonicity properties and comparative statics for this concept. In the present paper we simply assume nonemptiness of the equal treatment ε-core of games. With SGE along with per capita boundedness, for $\varepsilon > 0$ this assumption is satisfied for all sufficiently large games in parameterized collections. Such a result appears in [7,9]. Notice that we treat the equal treatment ε-core as a "stand-in" for the competitive equilibrium in the general context of the cooperative game theory. This motivates our use of the equal treatment ε-core and not the ε-core in the main subject of the present paper.

3 Laws of scarcity

A technical lemma is required. For $x, y \in \mathbf{R}^T$, let $x \cdot y$ denote the scalar product of x and y, i.e. $x \cdot y := \sum_{t=1}^{T} x_t y_t$.

Lemma. Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and, for $k = 1, 2$, let f^k

denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then

$$(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|.$$

Proof. Since (N, v) has β-effective B-bounded groups, there exists a partition $\left\{G^{1\ell}\right\}$ of S^1, such that $\left|G^{1\ell}\right| \leq B$ for any ℓ and $\sum_\ell v(G^{1\ell}) \geq v(S^1) - \beta \left\| f^1 \right\|$. Let us denote the profiles of $G^{1\ell}$ by g^ℓ. Observe that $\sum_\ell g^\ell = f^1$.

Since $f_t^2 \geq B$ for each t, it holds that $g^\ell \leq f^2$ for each ℓ. Therefore for each ℓ there exists a subset $G^{2\ell} \subset S^2$ with profile g^ℓ. Observe that since both $G^{1\ell}$ and $G^{2\ell}$ have profile g^ℓ, it holds that $\left|v(G^{1\ell}) - v(G^{2\ell})\right| \leq \delta \left\| g^\ell \right\|$. Since x^2 represents a payoff vector in the equal treatment ε-core of (S^2, v) and $G^{2\ell} \subset S^2$ has profile g^ℓ, the total payoff $x^2 \cdot g^\ell$ cannot be improved on by the coalition $G^{2\ell}$ by more than $\varepsilon \left\| g^\ell \right\|$. Thus, for each set $G^{2\ell} \subset S^2$ with profile g^ℓ, it holds that $x^2 \cdot g^\ell \geq v(G^{2\ell}) - \varepsilon \left\| g^\ell \right\| \geq v(G^{1\ell}) - (\varepsilon + \delta) \left\| g^\ell \right\|$. Adding these inequalities we have $x^2 \cdot f^1 \geq \sum_\ell v(G^{1\ell}) - (\varepsilon + \delta) \left\| f^1 \right\|$. It then follows that $x^2 \cdot f^1 \geq v(S^1) - (\varepsilon + \delta + \beta) \left\| f^1 \right\|$.

Since x^1 represents a payoff vector in the equal treatment ε-core of (S^1, v), $x^1 \cdot f^1$ is feasible for (S^1, v), that is, $x^1 \cdot f^1 \leq v(S^1)$. Combining these inequalities we have $(x^1 - x^2) \cdot f^1 \leq (\varepsilon + \delta + \beta) \left\| f^1 \right\|$. □

Now we can state and prove our main results.

3.1 Approximate cyclic monotonicity

We derive an exact bound on the amount by which an approximate core payoff vector for a given game can deviate from satisfying exact cyclic monotonicity. The bound depends on:

δ, the extent to which players within each of T types may differ from being exact substitutes for each other;

β, the maximal loss of per capita payoff from restricting effective coalitions to contain no more than B players; and

ε, a measure of the extent to which the ε-core differs from the core.

Our result is stated both for absolute numbers and for proportions of players of each type. If exact cyclic monotonicity were satisfied, then the right hand sides of the equations (1) and (2) below could both be set equal to zero.

Proposition 1. *Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), .., (S^K, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f_t^k \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then*

$$\begin{aligned}(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K \\ \leq (\varepsilon + \delta + \beta) \left\| f^1 + f^2 + .. + f^K \right\| \end{aligned} \quad (1)$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq K(\varepsilon + \delta + \beta). \tag{2}$$

That is, the equal treatment ε-core correspondence approximately satisfies cyclic monotonicity both in terms of numbers of players of each type and percentages of players of each type.

Proof. From Lemma we have $(x^k - x^{k+1}) \cdot f^k \leq (\varepsilon + \delta + \beta) \left\|f^k\right\|$ for $k = 1, .., K-1$ and $(x^K - x^1) \cdot f^K \leq (\varepsilon + \delta + \beta) \left\|f^K\right\|$. Summing these inequalities we get (1).

Alternatively we have $(x^k - x^{k+1}) \cdot \frac{f^k}{\|f^k\|} \leq (\varepsilon + \delta + \beta)$ for $k = 1, .., K-1$ and $(x^K - x^1) \cdot \frac{f^K}{\|f^K\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain (2). □

Remark. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\|.$$

This form of monotonicity is typically called simply *monotonicity* or *weak monotonicity*.

Note that weak monotonicity does not imply cyclic monotonicity.

Corollary. When $K = 2$, Proposition 1 implies that

$$(x^1 - x^2) \cdot (f^1 - f^2) \leq (\varepsilon + \delta + \beta) \left\|f^1 + f^2\right\|$$
$$\text{and} \quad (x^1 - x^2) \cdot \left(\frac{f^1}{\|f^1\|} - \frac{f^2}{\|f^2\|}\right) \leq 2(\varepsilon + \delta + \beta).$$

That is, the equal treatment ε-core correspondence is approximately monotonic.

Note that the bound of Proposition 1 and its Corollary holds for any partition of the player set into δ-substitutes.

3.2 Comparative statics

For $j = 1, .., T$ let us define $e^j \in \mathbf{R}^T$ such that $e^j_l = 1$ for $l = j$ and 0 otherwise. Our comparative statics results relate to changes in the abundances of players of a particular type.

Proposition 2. *Let (N, v) be in $\Gamma((\delta, T), (\beta, B))$ and let $(S^1, v), (S^2, v)$ be subgames of (N, v). Let $\{N[t]\}$ denote a partition of N into types and for each k let f^k denote the profile of S^k relative to $\{N[t]\}$. Assume that $f^k_t \geq B$ for each k and each t. For each k, let $x^k \in R^T$ represent a payoff vector in the equal treatment ε-core of (S^k, v). Then the following holds:*

(A) *If* $f^2 = f^1 + me^j$ *for some positive integer* m *(i.e., the second game has more players of approximate type* j *but the same numbers of players of other types) then*

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta) \frac{\|f^1 + f^2\|}{\|f^2 - f^1\|} = (\varepsilon + \delta + \beta) \frac{2\|f^2\| - m}{m}.$$

(B) *If* $\frac{f^2}{\|f^2\|} = (1-\mu)\frac{f^1}{\|f^1\|} + \mu e^j$ *for some* $\mu \in (0,1)$ *(i.e., the second game has proportionally more players of approximate type* j *but the same proportions between the numbers of players of other types) then*

$$(x_j^2 - x_j^1) \leq (\varepsilon + \delta + \beta) \frac{2-\mu}{\mu}.$$

That is, approximately the equal treatment ε*-core correspondence provides lower payoffs for players of a type that is more abundant.*

Proof. (A): Applying Corollary we get $(x^2 - x^1) \cdot me^j \leq (\varepsilon + \delta + \beta) \|f^1 + f^2\|$. Since $\|f^2\| = \|f^1\| + m$, this inequality implies our first result.

(B): From Lemma we have $(1-\mu)(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} \leq (1-\mu)(\varepsilon + \delta + \beta)$ and similarly $(x^2 - x^1) \cdot \frac{f^2}{\|f^2\|} \leq (\varepsilon + \delta + \beta)$. Summing these inequalities we obtain $(x^2 - x^1) \cdot (\frac{f^2}{\|f^2\|} - (1-\mu)\frac{f^1}{\|f^1\|}) \leq (2-\mu)(\varepsilon + \delta + \beta)$. Thus we get that $(x^2 - x^1) \cdot \mu e^j \leq (2-\mu)(\varepsilon + \delta + \beta)$. This inequality implies our second result. □

Obviously, again the bounds provided by the Proposition are independent of the specific partition of the player set into δ-substitutes. Note that all the bounds are exact; see Appendix.

4 Matching hospitals and interns: an example

Given the great importance of matching models (see, for example, Roth and Sotomayor [16] for an excellent study and numerous references to related papers), we present an application of our results to a model of matching interns and hospitals. Our example is highly stylized. For a more complete discussion of the matching interns and hospitals problem, we refer the reader to Roth [15].

The problem consists of the assignment of a set of interns $\mathcal{I} = \{1, .., i, .., I\}$ to hospitals. The set of hospitals is $\mathcal{H} = \{1, .., h, .., H\}$. The total player set N is given by $N = \mathcal{I} \bigcup \mathcal{H}$. Each hospital h has a preference ordering over the interns and a maximum number of interns $\overline{I}(h)$ that it wishes to employ. Interns also have preferences over hospitals. We'll assume $\overline{I}(h) \leq 9$ for all $h \in H$. This gives us a bound of 10 on the size of strictly effective groups ($\beta = 0$). For simplicity, we'll assume that both hospitals and interns can be ordered by the real numbers so that players with higher numbers in the ordering are more desirable. The rank held by a player will be referred to as the player's *quality*. More than one player may share the same rank in the ordering. In fact, we assume that the total payoff to a group consisting of a hospital and no more than nine interns is given by the sum of the rankings attached to the hospital and to the interns. Let us also assume that the rank assigned to any intern is between 0 and 1 and the rank assigned to any hospital is between 1 and 2. Thus, if the hospital is ranked 1.3 for example and is assigned 5 interns of quality .2 each, then the total payoff to that group is 2.3.

Since all interns have qualities in the interval $[0, 1)$ and similarly, all hospitals have qualities in the interval $[1, 2]$, given any positive real number $\delta = \frac{1}{n}$ for some positive integer n we can partition the interval $[0, 2]$ into $2n$ intervals, $[0, \frac{1}{n}), .., [\frac{j-1}{n}, \frac{j}{n}), .., [\frac{2n-1}{n}, 2]$, each of measure $\frac{1}{n}$. Assume that if there is a player with rank in the jth interval, then there are at least 10 players with ranks in the same interval.

Given $\varepsilon \geq 0$, let x^1 represent a payoff vector in the ε-core that treats all interns with ranks in the same interval equally and all hospitals with ranks in the same interval equally (that is, x^1 is equal treatment relative to the given partition of the total player set into types). Let us now increase the abundance of some type of intern that appears in N with rank in the jth interval for some j. We could imagine, for example, that some university training medical students increases the number of type j interns by admitting more students from another country. Let x^2 represent an equal treatment payoff vector in the ε-core after the increase in type j interns. It then holds, from result (A) of Proposition 2 that

$$(x_j^2 - x_j^1) \leq (\varepsilon + \frac{1}{n})\frac{\|f^1 + f^2\|}{\|f^2 - f^1\|}.$$

Of course this is not the most general application of our results – we could increase the proportions of players of one type by reducing the numbers of players of other types. Then part (B) of our Proposition could be applied.

It is remarkable that our results apply so easily. For this simple sort of example, it is probably the case that a sharper result can be obtained. This is beyond the scope of our current paper, however. Research in progress considers whether sharper results are obtainable with assortative matching of the kind illustrated by this example – that is, where players can be ordered so that players with higher ranks in the orderings are superior in terms of their marginal contributions to coalitions.

Finally, the parameter values that we have used in this example were chosen for convenience and simplicity. In principle, these could be estimated and various questions addressed. For example, are payoffs to interns approximately competitive? Do non-market characteristics such as ethnic background or gender make significant differences to payoffs?

5 Relationships to the literature and conclusions

Our results may be viewed as a contribution to the literature on comparative statics properties of solutions of games. As noted by Crawford [3], the first suggestion of the sort of results obtained in this paper may be in Shapley [18], who showed that in a linear optimal-assignment problem the marginal product of a player on one side of a market weakly decreases when another player is added to that side of the market and weakly increases when a player is added to the other side of the market. Kelso and Crawford [5], building on the model of Crawford and Knoer [4], show that, for a many-to-one matching market with firms and workers, adding one or more firms to the market makes the firm-optimal stable outcome weakly better for all workers and adding one or more workers makes the firm-optimal stable outcome weakly better for all firms. Crawford [3] extends these results to both sides of the market and to many-to-many matchings.[5] In contrast to this literature, our results are not restricted to matching markets and treat all outcomes in equal treatment ε-cores. Moreover, we demonstrate cyclic monotonicity. Instead of the assumptions of "substitutability" of Kelso and Crawford [5], however, we require our thickness condition and SGE. Unlike [5] and [3], our current results are limited to games with side payments – we plan to consider this limitation in future research.

Note that our results imply a certain *continuity* of comparative statics results with respect to changes in the descriptors of the total player set. In particular, the results are independent of the exact partition of players into approximate types. Specifically, given a number T of approximate types and a measure of the required closeness of the approximation, subject to the condition that players of each type are approximate substitutes for each other, our results apply independently of exactly where the boundary lines between types are drawn. Suppose, for example, that we wished to partition candidates for positions as hospital interns into three categories – say "good", "better" and "best". It may be that there is more than one way to partition the set of players into these categories while retaining the property that all players in each member of the partition are approximate substitutes for each other; the exact partition does not affect the results. Relating this feature of our work to general equilibrium theory, a finite set of commodities is typically considered to be an approximation to the real-world situation that all units of each commodity may differ. Descriptions of commodities are incomplete and a "commodity" is a group of objects that satisfy the description. For example, models of labor markets may have two types of workers, "skilled" and "unskilled" but no two workers (or two loaves of bread, or two oranges) may be exactly identical. In the differentiated commodities literature, results addressing this problem show that prices are continuous functions of attributes of commodities (cf., Mas-Colell [11]). Since our framework does not require a topology on the space of player types, continuity takes a different but valid form and is more directly apparent.

Besides the matching literature, our results are related to prior results obtained within the context of a pregame, cf. [23,26]. A pregame specifies a set of compact metric space of player types and a *single* worth function, assigning a worth to each finite list of attributes (repetitions allowed). Since there is only one worth function, all games derived from a pregame are related and, given the attributes of the members of a coalition, the payoff to that coalition is independent of the total player set in which the coalition is embedded; widespread externalities are not allowed. In contrast, our results apply to given games and, as in the earlier results for matching models, there is no requisite topological structure on the space of players types. While our results for a given game hold for all games in a collection

[5] And also to pair-wise stable outcomes but this is apparently not so directly related to our paper.

described by the same parameters, there are no necessary relationships between games. For example, consider the collection of games where two-player coalitions are effective and there are only two types of players. This collection includes two-sided assignment games, such as marriage games and buyer-seller games, and also games where *any* two-player coalition is effective. There appears to be no way in which one pregame can accommodate all the games in the collection. These considerations indicate that the framework of parameterized collections of games is significantly broader than that of a pregame.[6]

A major advantage of our approach over the prior approach using pregames is that, except for the special case of pregames satisfying strict small group effectiveness (or, in other words, 'exhaustion of gains to scale by coalitions bounded in size') with a finite number of exact types, *the conditions used in the prior literature cannot be verified* for any finite game.[7] That is, since the conditions are stated on the worth function of the entire pregame, which includes specification of the worths of arbitrarily large groups, or on the closeness of the worth function to the limiting per capita utility function, it is not possible to determine whether the conditions are satisfied. In contrast, given any game, values of parameters describing that game can be computed.[8]

Another major advantage of our approach is that we provide *exact bounds*, in terms of the parameters describing a game, on the amounts by which equal treatment ε-core payoff vectors can differ from satisfying cyclic monotonicity. We are unaware of any comparable results in the literature. The prior literature does not indicate the *sensitivity* of the results to specifications of bounds on group sizes and of types of players. Such an analysis is important for empirical testing since, in fact, few commodities are completely standardized. (This may be especially true in estimating hedonic prices.) Nor does the prior literature provide *empirically testable conclusions* on approximate monotonicity or comparative statics.

Numerous examples of games derived from pregames may lead one to expect our comparative statics result. Consider a glove game, for example where the payoff function can be written as $u(x, y) = \min\{x, y\}$. Suppose initially that the number of RH gloves, say x, is equal to the number of LH gloves, y, and both x and y are greater than one. Then the equal-treatment core can be described by the set $\{(p_x, p_y) \in \mathbb{R}^2_+ : p_x + p_y = 1\}$; each RH glove is assigned p_x and each LH glove is assigned p_y and a pair of gloves is assigned 1. Now increase the number of players with RH gloves. The equal treatment core is now described by $\{(0, 1)\}$; each RH glove is assigned 0 and each LH glove is assigned 1.

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[6] A short survey discussing parameterized collections of games and their relationships to pregames appears in [29].

[7] *Strict* small group efectiveness dictates that *all* gains to coalition formation can be realized by partitioning the total player set, no matter how large, into coalitions bounded in size. This condition was introduced in Wooders [23] (condition *) and, for NTU games, in Wooders [24], where it was called 'minimum efficient scale.'

[8] Since there may be many but a *finite* number of coalitions, in fact determining the required sizes of δ and T, β and B may be time-consuming but it is possible. In contrast, to verify that a pregame satisfies SGE or PCB requires consideration of an *infinite* number of payoff sets or, even more demanding, a limiting set of equal treatment payoffs.

In games with a finite set of player types, defining the core via linear programming also leads to a law of scarcity, quite immediately. Let (N, v) be a game with a finite number T of player types and with m_t players of type t, $t = 1, ..., T$. We take v as a mapping from subprofiles s of m ($s \in \mathbb{Z}_+^T$, $s \leq m$). Then, following Wooders [23], consider the following LP problem:[9]

$$\begin{aligned}&\text{minimize}_{p \geq 0}\;\; p \cdot m\\ &\text{subject to } p \cdot s \geq v(s) \text{ for all } s \leq m\end{aligned}$$

If the game has a nonempty core, then the solution p^* satisfies $v(m) = p^* \cdot m$. Now consider the same problem but with an increased number of players of type $\widehat{t}$ in the objective function for some $\widehat{t} \in \{1, ..., T\}$. Assume that the same inequalities are the only constraints; this imposes a form of strict small group effectiveness on the game – only groups with profiles $s \leq m$ are effective. It is clear that the payoff to players of type $\widehat{t}$ will not increase with the increase in the number of players of that type in the objective function since the constraint set has not changed – the payoff to type $\widehat{t}$ can only decrease. This suggests some of the initial intuition underlying comparative statics results for games. Under conditions roughly equivalent to those of Wooders [23] – that *all* gains to coalition formation can be exhausted by coalitions bounded in size – a proof of the comparative statics result and weak monotonicity of core payoff vectors was provided in [17]. We provide a more comprehensive discussion of the literature in [10].

6 Appendix

We construct some sequences of games to demonstrates that all the bounds we obtained in our results are *exact*, that is, the bound cannot be decreased.

I). Let us concentrate first on the central case $\delta = \beta = 0$. Consider a game (N, v) where any player can get only 1 unit or less in any coalition and there are no gains to forming coalitions. This game has strictly effective 1-bounded groups and all agents are identical. Formally, however, we may partition the set of players into many types. Thus $(N, v) \in \Gamma((0, \tau), (0, 1))$ for any integer τ, $1 \leq \tau \leq |N|$. Notice also that for any $\varepsilon \geq 0$ the ε-core of the game is nonempty and very simple: it includes all payoff vectors that are feasible and provide at least $1 - \varepsilon$ for each of the players. All the games that we are going to construct will be subgames of a game (N, v).

a). For the bound in Lemma we can present even a single game with two payoffs vectors that realize this bound. Namely, let $\tau = 1$ (all players are of one type) and let us consider any two subgames S^1, S^2 with the same number of players and the equal treatment payoffs $x^1 = 1$ and $x^2 = 1 - \varepsilon$. Then $(x^1 - x^2) \cdot f^1 = \varepsilon \left\| f^1 \right\|$.

b). For the bound in Proposition 1, for $K \leq |N|$ and some nonnegative integer $l \leq |N| - K$, let us consider $\tau = K$ and the subgroups $S^1, .., S^K$ with the profiles

[9] The core has been described as an outcome of a linear programming problem since the seminal works of Gilles and Shapley. Wooders [23] introduces the linear programming formulation with player types (see also [26]).

$f^1, .., f^K$ where $f_t^k = l + 1$ for $t = k$ and 1 otherwise. Let also consider payoff vectors x^k where $x_t^k = 1$ for $t = k$ and $1 - \varepsilon$ otherwise. Then $(x^i - x^j) \cdot f^i = \varepsilon l$ for any $i \neq j$. Hence

$$\begin{aligned}&(x^1 - x^2) \cdot f^1 + (x^2 - x^3) \cdot f^2 + .. + (x^K - x^1) \cdot f^K \\ &= \varepsilon l K = \varepsilon \left\| f^1 + f^2 + .. + f^K \right\| \frac{l}{l + K}\end{aligned}$$

and

$$(x^1 - x^2) \cdot \frac{f^1}{\|f^1\|} + (x^2 - x^3) \cdot \frac{f^2}{\|f^2\|} + .. + (x^K - x^1) \cdot \frac{f^K}{\|f^K\|} = K\varepsilon \frac{l}{l + K}.$$

It is straightforward to verify that for any fixed K both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

c). For the bound in Proposition 2 it is enough to concentrate on (A) since it is a special case of the result (B). For $|N| \geq 2$ let us consider $\tau = 2$ and $l \leq |N| - 2$. Then consider the subgroups S^1, S^2 with the profiles $f^1 = (1, 1)$ and $f^2 = (l+1, 1)$ and payoff vectors $x^1 = (1 - \varepsilon, 1)$ and $x^2 = (1, 1)$. Then

$$(x_1^2 - x_1^1) = \varepsilon = \varepsilon \frac{\left\| f^1 + f^2 \right\|}{\left\| f^2 - f^1 \right\|} \frac{l}{l + 4}.$$

It follows that both our bounds in Proposition 2 can not be improved for sequences of games (N, v), with $|N|$ going to infinity, for subgames constructed as above with l going to infinity.

II). It is easy to modify our example to allow for non-zero δ and β in a such a way that we will have the same profiles as in Part I, but will use the payoffs of $1 + \delta + \beta$ and $1 - \varepsilon$ instead of 1 and $1 - \varepsilon$. This will lead us to the appearance of $\varepsilon + \delta + \beta$ on the places of ε in all bound in Part I. We leave it as a simple exercise for the interested reader.

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EQUIVALENCE OF GAMES AND MARKETS[1]

BY MYRNA HOLTZ WOODERS[2]

The author proves an equivalence between large games with effective small groups of players and games generated by markets. Small groups are effective if all or almost all gains to collective activities can be achieved by groups bounded in size. A market is an exchange economy where all participants have concave, quasi-linear payoff functions. The market approximating a game is socially homogeneous—all participants have the same monotonic nondecreasing, and 1-homogeneous payoff function. Our results imply that any market (more generally, any economy with effective small groups) can be approximated by a socially homogeneous market.

KEYWORDS: Games, markets, market-game equivalence, small group effectiveness, social homogeneity.

1. SMALL GROUP EFFECTIVENESS AND SOCIALLY HOMOGENEOUS MARKETS

THIS PAPER ESTABLISHES AN EQUIVALENCE between socially homogeneous markets with payoff-constant returns and large games with effective small groups of players. Small groups are effective if all or almost all gains to collective activities can be realized by the activities of groups of players bounded in absolute size. A market is defined as a private-goods economy where all participants have concave payoff functions that are linear in money. The market is socially homogeneous if all participants have the same payoff function; the market satisfies payoff-constant returns if the payoff functions are 1-homogeneous. The market approximating a large game with effective small groups also has the property that the payoff function is continuous. When the continuity assumption is relaxed at the boundaries of the commodity space, an equivalence of large games and markets holds with only the apparently mild requirement of per capita boundedness of payoffs.[3] The economies to which the results apply include ones with nonmonotonicities, nonconvexities, and consumption sets unbounded from above and below. Also, the economies may have public goods,

[1] Previous versions of this paper include parts of *C.O.R.E* Discussion Paper No. 8842, "Large Games are Market Games" (1988) and University of Toronto Department of Economics Working Paper 1904 "Equivalence of Perfect Competition and Effective Small Groups" (1992).

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[3] The condition of per capita boundedness was introduced in the study of large games in Wooders (1979); see Kannai (1992) or Wooders (1992b) for a discussion of the model and the main result. Small group effectiveness was introduced in Wooders (1992a).

collectively consumed and produced goods, and other deviations from the Arrow-Debreu model of an exchange economy.

Our results are obtained in the framework of a model of games with a finite set of player types and side payments. A game consists of a set of players and a function, called the characteristic function, which assigns an amount of surplus to each group of players. The characteristic function is intended to provide an abstract summary description of economic data and is not intended to describe economic behavior; there is no presumption of cooperation.

Our first market-game equivalence result states that the class of large games with effective small groups and the class of continuous, socially homogeneous markets with payoff-constant returns are approximately equivalent. Our second market-game equivalence result establishes that per capita boundedness suffices to ensure the equivalence of large games and socially homogeneous markets with payoff-constant returns. Note that with only per-capita boundedness, the markets are not required to be continuous. Since games generated by markets satisfy per capita boundedness, our result establishes that large markets are approximated by socially homogeneous markets with payoff-constant returns. We also establish that with thickness of the player set, bounding the percentages of players of each type away from zero, per capita boundedness and small group effectiveness are equivalent, thus relating our two conditions and our two market-game equivalence results.

Since the conditions of continuity, social homogeneity, and payoff-constant returns appear restrictive, our first market-game equivalence result indicates the power of the condition of small group effectiveness. Since per capita boundedness simply bounds average payoffs away from infinity and thus clearly permits a variety of economic structures, our second market-game equivalence result indicates the broad applicability of small group effectiveness.

In the approach of this author the set of marketed commodities is viewed as endogenous. There may be many different definitions of commodities for a market approximating a game. For example, for a game derived from an economy, the commodities in an approximating market can be chosen to be the same as those specified for the economy.[4] In this paper it is established that a large game or a large economy, regardless of the number of types of commodities, is approximated by a market where the number of types of commodities is no larger than the number of types of participants.

The asymptotic equivalence of markets and economies with effective small groups is suggested by a number of other results. One such suggestion was made by Tiebout (1956), who argued that economies with local public goods are "market-like." More recently several papers have shown that cores of such economies converge to price taking equilibrium outcomes.[5] Moreover, core convergence holds in exchange economies even when coalitions are constrained

[4] An application using this approach appears in Wooders (1992c).

[5] See also Buchanan (1965). Results on convergence of cores in economies with public goods are reviewed in Wooders (1994).

to be small relative to the total economy.[6] Recall that Shapley and Shubik (1969) showed that large exchange economies have nonempty approximate cores. Wooders (1983) shows that large games satisfying per capita boundedness and with a finite number of types have nonempty approximate cores and Kaneko and Wooders (1986) show that continuum games with small (finite) coalitions have nonempty cores. Shapley and Shubik (1966) showed an equivalence between the class of totally balanced games, those with the property that the game and every subgame have nonempty cores, and the class of games derived from continuous markets. The nonemptiness of approximate cores of large games implies approximate balancedness and thus implies that large games are approximately market games. Shapley (1964) first showed that Shapley values of replicated exchange economies converge to core payoffs. With a strong form of small group effectiveness (boundedness of individual marginal contributions to coalitions), Wooders and Zame (1987) show that Shapley values of large games are in approximate cores.[7] Moreover, the following "market-like" properties of large games with effective small groups are shown in Wooders (1992b): approximate cores of large games converge if and only if small groups are effective; approximate cores of large games with small effective groups satisfy a game-theoretic analogue of the "Law of Demand" (an increase in the percentage of players of any type does not cause an increase and may cause a decrease in core payoffs to each player of that type); approximate cores of large games with effective small groups and many players of each type have an approximate equal-treatment property—a core payoff assigns most players of the same type approximately the same payoff.[8]

The research noted above raises the possibility not only that large games and economies with effective small groups *behave* like competitive markets but also that these games and economies *are* competitive markets or close to them. Such observations suggest our equivalence result, which explains the similarities between markets and games.

Some relaxation of the finite-type and side payment assumptions of this paper is possible. Using familiar techniques of approximating a compact metric space of types/attributes by finite sets of types, extensions of all our results can be obtained with a compact metric space of player types and with differentiated commodity markets, as in Mas-Colell (1975) or Jones (1984), for example.[9] Indeed, no topological structure on the set of player types is necessary to make apparent the closeness between a large game with effective small groups and a

[6] See Anderson (1992) for a survey including research on convergence of the core with restrictions on coalition size.

[7] We conjecture that this result holds under the milder condition of small group effectiveness. Wooders and Zame (1987) provide further references to related literature on the Shapley value of large games and economies.

[8] Further references to related literature are provided in Wooders (1992b).

[9] In fact, the first version of this paper, C.O.R.E Discussion Paper No. 8842 (1988), used a compact metric space of player types. The framework with a compact metric space of player types and the assumption of small group effectiveness is further studied in Wooders (1992, 1993b). The later paper contains some extensions of the results of this paper.

market game.[10] The formulation with a finite number of types of players, however, is fundamental and permits a number of other questions to be addressed. An advantage of the formulation with a metric space of attributes (either finite or infinite) is that it facilitates the study of the continuum limit. The restriction to games with side payments can also be relaxed and some analogues of our results still obtain.[11]

The results in this paper obviously apply to the special case of games and economies with bounded effective group sizes, those when all gains to improvement can be realized by strict subgroups of the population. Games with bounded effective group sizes and a finite number of types are basic to the research of this paper and other related works since they are especially tractable and since they approximate games with effective small groups. Some special properties of games with bounded effective group sizes include that for all sufficiently large games, all core payoffs have the equal-treatment property (Wooders (1983, Theorem 3)). For a discussion of the special properties of games with bounded effective group sizes and a unified presentation of a number of results, see Wooders (1992b).

It should be observed that we are not concerned here with game-theoretic justification for the price-taking hypothesis of perfect competition. Rather than showing any equivalence of outcomes of solution concepts, we establish a game-theoretic equivalence between large games and large markets themselves. We return to this in the concluding section.

2. GAMES AND PREGAMES

There is a given finite number T of types of players. Let Z_+^T denote the T-fold Cartesian product of the nonnegative integers. A *profile* $f = (f_1, \dots, f_T) \in Z_+^T$ is a group of players, described by the number f_t of players of each type t in the group. The profile describing a group consisting of only one player of type t and no other players is denoted by χ_t. Note that if f is a profile and r is a positive integer, then rf is a profile. Also, if f and g are profiles, then so is $f+g$. Given a profile f, define $\|f\| = \Sigma_t f_t$, called the *norm* of f; the norm of f is the total number of players in the group f. The *support* of a profile f is the set $\{t \in \{1, \dots, T\}\colon f_t \neq 0\}$. A *partition of a profile* f is a collection of profiles (f^k), not necessarily all distinct, satisfying $\Sigma_k f^k = f$. A partition of a profile is analogous to a partition of a set except that all members of a partition of a set are distinct.

Let Ψ be a function from the set Z_+^T of profiles to $\boldsymbol{R}_+$ with $\Psi(0) = 0$. The pair (T, Ψ) is called a *pregame* with *characteristic function* Ψ. The value $\Psi(f)$ is the total payoff a group of players f can achieve by collective activities of the

[10] See Wooders (1993a).

[11] See Wooders (1983) for the model of games without side payments, some relevant results, and Wooders (1991) for further results and references.

group membership. The pregame is *superadditive* if

$$(2.1) \qquad \Psi(f) = \max \sum_{g \in P} \Psi(g),$$

where P is a partition of f and the maximum is taken over all partitions of f.

A *game determined by a pregame* (T, Ψ), called simply a *game* or a *game in characteristic form*, is a pair $[n, \Psi]$ where n is a profile, interpreted as a description of the total player set in the game, and the characteristic function Ψ is restricted to subprofiles of n. Note that a pregame specifies payoffs for every profile whereas a game specifies a total population, and the only relevant profiles are those no larger than the population. A payoff for the game is a vector x in $\boldsymbol{R}^T$. A payoff x is *feasible* if $x \cdot n \leqslant \Psi(n)$, and it is *Pareto optimal* if $x \cdot n = \Psi(n)$.

A pregame (T, Ψ) satisfies *small group effectiveness* if it is superadditive and if, given any real number $\varepsilon > 0$, there is an integer $\eta_0(\varepsilon)$ such that for each profile f, for some partition (f^k) of f:

$$(2.2) \qquad \|f^k\| \leqslant \eta_0(\varepsilon) \qquad \text{for each subprofile } f^k \text{ in the partition, and}$$

$$(2.3) \qquad \Psi(f) - \sum_k \Psi(f^k) \leqslant \varepsilon \|f\|;$$

for every profile f, almost all (within ε per capita) of the gains to collective activities can be realized by aggregating collective activities within group of participants bounded in absolute size.[12]

When a pregame satisfies small group effectiveness we say that it has *effective small groups*. Note that this condition does not rule out effective large groups, although it does imply that only smaller and smaller increases in per capita payoff can be achieved by the formation of larger and larger groups. Elsewhere the condition has been called "inessentiality of large groups;" collective activities of large groups are not essential for the realization of almost all gains to group formation. The term "group" is used instead of the more commonly used term "coalition" since "coalition" may suggest cooperative behavior. Small group effectiveness is intended to be a condition on economic and game-theoretic primitives rather than on behavior.[13]

3. MARKETS AND PREMARKETS

A premarket consists of a finite number of commodity types and of participant types. A participant type is described by two characteristics, a payoff function and an endowment. Formally, a *premarket* is a pair $(\boldsymbol{R}_+^M, A)$ where $\boldsymbol{R}_+^M$

[12] Note that the profile f in the definition may be arbitrarily large relative to the bound $\eta_0(\varepsilon)$. Thus, in "very large" coalitions, almost all gains to collective activities can be realized as the aggregate of activities of "negligible" groups of participants. This suggests the f-core approach, introduced in Kaneko and Wooders (1986), where finite coalitions are effective in games with a continuum of players.

[13] The condition of small group effectiveness is less restrictive than boundedness of individual marginal contributions to coalitions, as in Wooders and Zame (1987) for example.

is the *commodity space* and A is the *set of types of participants*. A *commodity bundle* x is a vector $x \in \boldsymbol{R}_+^M$. The set $A = \{a^i\colon i = 1, \dots, I\}$ is a finite indexed collection of triples, $a^i = (\omega^i, u^i)$, where each ω^i is a commodity bundle, called the *endowment* of a participant of type i, and each u^i is a concave, monotonic nondecreasing function from the set of commodity bundles $\boldsymbol{R}_+^M$ to the reals, called the *payoff function* of a participant of type i.[14] Assume that payoff functions are normalized so that $u^i(\omega^i) \geqslant 0$ for all i. The premarket is *continuous* if the payoff functions of all participants are continuous and the premarket satisfies *payoff-constant returns* if the payoff functions of all participants are 1-homogeneous. The premarket $(\boldsymbol{R}_+^M, A)$ is *socially homogeneous* if there is a function u such that for all participants i, $u^i = u$.

A *market* determined by a premarket $(\boldsymbol{R}^M, A)$ is a pair $[n, A]$ where n is the *market population profile*, a vector in Z_+^I listing the number of participants with attributes a^i for each $i = 1, \dots, I$. We refer to participants with the same endowments and payoff functions as participants of the same "type."

REMARK 1: Socially homogeneous markets enjoy particularly pleasing properties. Since all participants have the same concave payoff function, competitive prices are determined by the subgradients of the payoff function. For almost all distributions of total endowments the competitive price system is uniquely determined (up to a normalization). Moreover, from these observations and from convergence of approximate cores to limiting competitive payoffs it follows that cores and in a certain sense, approximate cores, of large socially homogeneous economies are typically small. Since large games with effective small groups are approximately markets, they inherit these properties. See Wooders (1992b) for further discussion.

4. ASYMPTOTIC MARKET-GAME EQUIVALENCE

We establish an approximate equivalence between large economies with effective small groups, modeled as abstract games, and continuous, socially homogeneous markets with payoff-constant returns. Proofs are given in Section 7.

Let $(\boldsymbol{R}_+^M, A)$ be a premarket with $|A| = I$. For each profile $f = (f_1, \dots, f_I) \in Z_+^I$, define $\Lambda(f)$ by

$$(4.1) \qquad \Lambda(f) = \max \sum_{i=1}^{I} \sum_{j=1}^{f_i} u^i(x^{ij}),$$

where the maximum is taken over the set of commodity bundles $\{x^{ij}\}$ satisfying

$$(4.2) \qquad \sum_{i=1}^{I} \sum_{j=1}^{f_i} x^{ij} = \sum_{i=1}^{I} f_i \omega^i.$$

[14] For interpretation, we may think of the payoff function of trader i as defined by $U^i(x, \xi) = u^i(x) + \xi$, where ξ is a real number, to be thought of as "final money balance." The money plays no formal role in this paper.

The pair (I, Λ) is *the pregame induced by the premarket* $(\boldsymbol{R}_+^M, A)$. Any pregame which can be induced by some premarket is a *market pregame*. We say that *a premarket satisfies small group effectiveness* if its induced pregame satisfies small group effectiveness.

Let (T, Ψ) and $(\hat{T}, \hat{\Psi})$ be two pregames. The pregames (T, Ψ) and $(T, \hat{\Psi})$ are *asymptotically equivalent* if $\hat{T} = T$ and if, given any real number $\varepsilon > 0$, there is an integer $\eta_1(\varepsilon)$ such that for all profiles f with $\|f\| \geqslant \eta_1(\varepsilon)$

$$(4.3) \qquad \left|\hat{\Psi}(f) - \Psi(f)\right| \leqslant \varepsilon\|f\|.$$

Asymptotic equivalence of two pregames implies that for any large profile the per capita payoffs assignable to that profile by the two characteristic functions are approximately equal. A pregame (T, Ψ) and a premarket $(\boldsymbol{R}_+^M, A)$ are said to be *asymptotically equivalent* if the pregame is asymptotically equivalent to the pregame determined by the premarket. This requires, of course, that the number $|A|$ of types of participants in the premarket equals the number of types T of players in the pregame. The following two Theorems lead to our first market-game equivalence result.

THEOREM 1: *A continuous, socially homogeneous premarket with payoff-constant returns satisfies small group effectiveness.*

THEOREM 2: *A pregame (T, Ψ) has effective small groups if and only if there is an asymptotically equivalent, continuous, and socially homogeneous premarket $(\boldsymbol{R}_+^M, A)$ satisfying payoff-constant returns.*

Theorem 1 states that continuous, socially homogeneous market pregames satisfy small group effectiveness. Theorem 2 states that the class of pregames with effective small groups is asymptotically equivalent to the class of continuous, socially homogeneous market pregames satisfying payoff-constant returns. The following conclusion is the main result of this paper.

THEOREM 3 (Market-Game Equivalence): *The class of continuous, socially homogeneous premarkets with payoff-constant returns and the class of pregames with effective small groups are asymptotically equivalent.*

A market with many participants is the basic model of a perfectly competitive market. Among other properties, large markets satisfy nonemptiness of the core and convergence of the core to the Walrasian equilibrium payoffs. Our result shows that large games with effective small groups are asymptotically equivalent to market games. This implies that for any large game with effective small groups, a Walrasian outcome of an approximating market is an approximately feasible payoff for the game. Therefore, for the game, there are Pareto optimal payoffs that are "close" to Walrasian payoffs of an approximating market. Since these payoffs are close to Walrasian payoffs we can regard them as "approximate competitive equilibrium" payoffs. When the number of participants is large, economies with effective small groups have approximate competitive outcomes; there is some set of commodities and a price system for these commodities that constitutes an approximate competitive equilibrium. This remark holds even for economies with public goods, as in Wooders (1980) for example.

REMARK 2: The condition of payoff-constant returns in Theorem 1 is included for symmetry with the conditions of Theorem 2; it is not required for the conclusion of the Theorem. In the next section, we show that provided the percentages of players of each type are bounded away from zero, without any additional assumptions market pregames satisfy small group effectiveness. The condition of social homogeneity in Theorem 1 is important in limiting the effects of "scarce" player types, those appearing in the total player set in vanishingly small proportions.

REMARK 3: Small group effectiveness is a natural and powerful condition, and applies to diverse economies. As we show in the next section, if we ignore boundaries of the commodity space, small group effectiveness is equivalent to per capita boundedness. Thus, the more restrictive the conditions on the premarkets approximating pregames with effective small groups, the more striking the result. In the above theorems the conditions on the premarkets are chosen to be restrictive. It can be demonstrated that if either continuity of social homogeneity is relaxed, Theorem 2 no longer holds. Our results in this paper imply that any premarket with effective small groups is asymptotically equivalent to a socially homogeneous premarket. Other conditions ensuring asymptotic social homogeneity of markets and economies more generally are discussed in Wooders (1993b).

5. PER CAPITA BOUNDEDNESS AND THICK MARKETS

Collections of profiles with the property that the percentages of players of each type are bounded away from zero are called "thick." Formally, let $\rho > 0$ be a real number and define

$$P(\rho) = \left\{ f \in Z_+^T\text{: for each } t = 1, \ldots, T \text{ either } \left(\frac{1}{\|f\|}\right) f_t > \rho \right.$$
$$\left. \text{or } \left(\frac{1}{\|f\|}\right) f_t = 0 \right\}.$$

The set $P(\rho)$ is called the set of *ρ-thick* profiles.

A pregame (T, Ψ) satisfies *per capita boundedness* if there is a constant A such that for all profiles f it holds that

$$\frac{\Psi(f)}{\|f\|} < A. \tag{5.1}$$

The per capita boundedness condition was introduced in Wooders (1979, 1983).

When profiles are required to be thick, per capita boundedness is equivalent to small group effectiveness.

THEOREM 4: *Let (T, Ψ) be a pregame and let $\varepsilon > 0$ and $\rho > 0$ be given positive real numbers. Then*

(5.2) *there is an integer $\eta_1(\varepsilon, \rho)$ such that:*

for each profile f in $P(\rho)$, there is a partition (f^k) of f with

$\|f^k\| \leqslant \eta_1(\varepsilon, \rho)$ *for each subprofile f^k in the partition and*

$$\Psi(f) - \sum_k \Psi(f^k) \leqslant \varepsilon \|f\|$$

if and only if there is a constant A such that (5.1) *holds for all f in $P(\rho)$.*

Let (T, Ψ) and $(\hat{T}, \hat{\Psi})$ be two pregames. The two pregames are *asymptotically quasi-equivalent* if $T = \hat{T}$ and if for each $\rho > 0$ (4.3) holds for all ρ-thick profiles, that is,

given any positive real numbers $\varepsilon > 0$ and $\rho > 0$ there is an integer $\eta_2(\varepsilon, \rho)$ such that for all profiles f in $P(\rho)$ with $\|f\| \geqslant \eta_2(\varepsilon, \rho)$,

$$|\Psi(f) - \hat{\Psi}(f)| \leqslant \varepsilon \|f\|.$$

We say *that a premarket is asymptotically quasi-equivalent to a pregame* if the pregame derived from the premarket is asymptotically quasi-equivalent to the pregame. Also, *two premarkets are asymptotically quasi-equivalent* if their induced pregames are asymptotically quasi-equivalent.

The next two Theorems are analogues of Theorems 1 and 2.

THEOREM 5: *A market pregame satisfies per capita boundedness.*

THEOREM 6: *A pregame (T, Ψ) satisfies per capita boundedness if and only if there is a socially homogeneous premarket $(\boldsymbol{R}_+^M, A)$ with payoff-constant returns such that (T, Ψ) and $(\boldsymbol{R}_+^M, A)$ are asymptotically quasi-equivalent.*

Theorems 5 and 6 illustrate that besides ensuring boundedness of per capita payoffs, the role of small group effectiveness in our results is to ensure that arbitrarily small percentages of players of "scarce types" cannot have significant effects on per capita payoffs of large groups. Our second market-game equivalence result follows.

THEOREM 7 (Quasi-Equivalence of Games Satisfying Per Capita Boundedness and Socially Homogeneous Markets): *The class of socially homogeneous premarkets with payoff-constant returns and the class of pregames satisfying per capita boundedness are asymptotically quasi-equivalent.*

REMARK 4: With a finite number of types of participants (or of commodities), the condition of per capita boundedness appears extremely mild and easy to apply. Moreover, similar conditions have a long history in economy theory. Small group effectiveness has the advantage that besides ensuring that there are continuous approximating markets, it is applicable to situations where all players/commodities may differ, for example, situations with a compact metric space of attributes (Wooders (1992a, 1993b), for example) and ones with no topology on the space of player types (Wooders (1993c)).

6. THE NUMBER OF COMMODITIES

To show that a pregame with effective small groups is asymptotically equivalent to a premarket a particular premarket is constructed. This premarket is itself of interest as it has a natural interpretation and permits the statement of another result. In the premarket constructed the commodities are the players and the derived markets can be interpreted as ones where players hire the participation of other players in groups. This interpretation is especially suited to economies with production, economies with clubs, labor markets, and attribute games in general, where, as in the cost allocation literature, the players of a game are interpreted as commodities or as attributes of economic participants. The payoff function is constructed as follows. Let (T, Ψ) be a pregame with effective small groups. Define a function u by

$$(6.1) \qquad u(x) = \|x\| \lim_{\nu \to \infty} \frac{\Psi(f^\nu)}{\|f^\nu\|},$$

where $\{f^\nu\}$ is any sequence of profiles with the properties that $\|x\| \lim_{\nu \to \infty} (1/\|f^\nu\|) f^\nu = x$. Provided that the pregame (T, Ψ) satisfies small group effectiveness the function u is well-defined. Our final Theorem establishes that *whatever* the number of types of commodities in a premarket, the premarket can be approximated by another premarket with the number of commodities no larger than the number of types of participants and the payoff function of all the participants can be taken as that given by (6.1).

THEOREM 8: *Let $(\boldsymbol{R}_+^M, A)$ be a premarket satisfying small group effectiveness and let $|A| = I$. Let (I, Λ) denote the derived pregame. Then there is an asymptotically equivalent socially homogeneous premarket $(\boldsymbol{R}_+^{M'}, A')$ satisfying payoff-constant returns and with the number of commodities M' equal to the number I of types of players in the premarket $(\boldsymbol{R}_+^M, A)$.*

An analogous result holds for premarkets and asymptotically quasi-equivalent premarkets.

7. PROOFS OF THE THEOREMS

Small group effectiveness dictates that all or almost all gains to collective activities can be realized by groups bounded in absolute size. This implies that, in a certain sense, the games can be approximated by ones with bounded effective group sizes. Most results on large games with effective small groups referenced in this paper were first proven for games satisfying the stronger requirement. We proceed here also by first proving a result for games with bounded effective group sizes.

Let (T, Ψ) be a pregame. The pregame satisfies *bounded effective group sizes* if there is an integer B such that for each profile f there is a partition (f^k) of f satisfying

$$(7.1)\qquad \|f^k\| \leqslant B \text{ for each } k \quad \text{and}$$

$$(7.2)\qquad \Psi(f) - \sum_k \Psi(f^k) = 0;$$

all gains to collective activities can be realized by groups bounded in size by B.

We will use the observation that from concavity, the value of the maximum in (4.1) is unchanged when the allocation $\{x^{ij}\}$ is required to have the equal-treatment property, that is, for each i and for all j and j', $x^{ij} = x^{ij'}$.

LEMMA 1: *Let (T, Ψ) be a superadditive pregame with bounded effective group sizes. Let $\{f^\nu\}$ and $\{g^\nu\}$ be sequences of profiles with the properties that*

$$(7.3)\qquad \|f^\nu\| \to \infty \quad \textit{and} \quad \|g^\nu\| \to \infty \textit{ as } \nu \textit{ becomes large} \quad \textit{and}$$

$$(7.4)\qquad \lim_{\nu\to\infty}\left(\frac{1}{\|f^\nu\|}\right) f^\nu = \lim_{\nu\to\infty}\left(\frac{1}{\|g^\nu\|}\right) g^\nu.$$

Then

$$(7.5)\qquad \lim_{\nu\to\infty}\left(\frac{1}{\|f^\nu\|}\right)\Psi(f^\nu) = \lim_{\nu\to\infty}\left(\frac{1}{\|g^\nu\|}\right)\Psi(g^\nu).$$

PROOF OF LEMMA 1: We prove the result by contradiction. Suppose that (T, Ψ), $\{f^\nu\}$, and $\{g^\nu\}$ satisfy the conditions of the Lemma. Let B denote a bound on effective group sizes. Define $A = \max \Psi(f)$, where the maximum is taken over all profiles f with $\|f\| \leqslant B$. Since $(\Psi(h)/\|h\|) \leqslant A$ for all profiles h it holds that the sequence $\{\Psi(f^\nu)/\|f^\nu\}$ has a converging subsequence. Suppose the sequences $\{\Psi(f^\nu)/\|f^\nu\|\}$ and $\{\Psi(g^\nu)/\|g^\nu\|\}$ both converge and there is a positive real number $\delta > 0$ such that

$$(7.6)\qquad \lim_{\nu\to\infty}\frac{\Psi(f^\nu)}{\|f^\nu\|} > \lim_{\nu\to\infty}\frac{\Psi(g^\nu)}{\|g^\nu\|} + 2\delta.$$

Define $f := \lim_{\nu\to\infty}(1/\|f^\nu\|)f^\nu = \lim_{\nu\to\infty}(1/\|g^\nu\|)g^\nu$. Define the sequences of profiles $\{h^\nu\}$ and $\{l^\nu\}$ by

$$(7.7)\qquad h_t^\nu = f_t^\nu \quad \text{if} \quad f_t \neq 0,$$

$$(7.8)\qquad h_t^\nu = 0 \quad \text{if} \quad f_t = 0, \quad \text{and}$$

$$(7.9)\qquad l_t^\nu = f_t^\nu - h_t^\nu.$$

From boundedness of effective group sizes it follows that

$$\Psi(f^\nu) - \Psi(h^\nu) \leqslant AB\|l^\nu\|$$

and therefore, since $(\|l^\nu\|/\|f^\nu\|) \to 0$ as $\nu \to \infty$, $\lim_{\nu\to\infty}(\Psi(f^\nu)/\|f^\nu\|) = \lim_{\nu\to\infty}(\Psi(h^\nu)/\|h^\nu\|)$.

Define $L := \lim_{\nu\to\infty}(\Psi(f^\nu)/\|f^\nu\|)$. Let ν_0 be sufficiently large so that for all $\nu \geqslant \nu_0$ it holds that

$$\frac{\Psi(h^\nu)}{\|h^\nu\|} \geqslant L - \delta. \tag{7.10}$$

From (7.4) we can suppose without loss of generality that the support of h^ν is contained in the support of $g^{\nu'}$ for all ν and ν' (that is, $h_t^\nu \neq 0$ implies $g_t^\nu \neq 0$). It follows that there is an integer $\nu_1 \geqslant \nu_0$ such that for all sufficiently large terms ν, for some integer r_ν,

$$g^\nu \geqslant r_\nu h^{\nu_1} \quad \text{and}$$

$$\frac{r_\nu\|h^{\nu_1}\|}{\|g^\nu\|} \geqslant \frac{L-2\delta}{L-\delta}, \tag{7.11}$$

that is, g^ν permits r_ν copies of h^{ν_1}, with only a small percentage of "leftovers." From superadditivity, (7.10) and (7.11),

$$\frac{\Psi(g^\nu)}{\|g^\nu\|} \geqslant \frac{r_\nu \Psi(h^{\nu_1})}{r_\nu\|h^{\nu_1}\|} \cdot \frac{r_\nu\|h^{\nu_1}\|}{\|g^\nu\|} \geqslant (L - 2\delta) = \lim_{\nu\to\infty} \frac{\Psi(f^\nu)}{\|f^\nu\|} - 2\delta,$$

for all $\nu \geqslant \nu_1$, a contradiction to (7.6).

The result now follows from the fact per capita payoffs are bounded. Thus, in general, the sequences $\{\Psi(f^\nu)/\|f^\nu\|\}$ and $\{\Psi(g^\nu)/\|g^\nu\|\}$ have converging subsequences. Since any pair of converging subsequences converge to the same limit both sequences must converge to the same limit. *Q.E.D.*

LEMMA 2: *Let (T, Ψ) be a pregame satisfying small groups effectiveness. Let $\{f^\nu\}$ and $\{g^\nu\}$ be sequences of profiles satisfying* (7.3) *and* (7.4). *Then* (7.5) *holds.*

PROOF OF LEMMA 2: We prove the result by contradiction. Suppose that (T, Ψ), $\{f^\nu\}$, and $\{g^\nu\}$ satisfy the conditions of the Lemma but not (7.5). Since small group effectiveness implies per capita boundedness (which is easy to show), we can suppose without loss of generality that the sequences $\{(1/\|f^\nu\|)\Psi(f^\nu)\}$ and $\{(1/\|g^\nu\|)\Psi(g^\nu)\}$ both converge and that for some positive real number $\varepsilon_0 > 0$, for all terms f^ν and g^ν in the sequences,

$$\left|\left(\frac{1}{\|f^\nu\|}\right)\Psi(f^\nu) - \left(\frac{1}{\|g^\nu\|}\right)\Psi(g^\nu)\right| > 3\varepsilon_0. \tag{7.12}$$

From small group effectiveness we may choose an integer B with the property that for each profile h there is a partition (h^k) of h satisfying

$$\|h^k\| \leqslant B \quad \text{for each } k \text{ and}$$

$$\Psi(h) - \sum_k \Psi(h^k) \leqslant \varepsilon_0\|h\|.$$

We next construct a pregame with effective group sizes bounded by B. For each profile f define $\Gamma(f)$ by

$$\Gamma(f)=\max \sum_k \Psi(f^k),$$

where the maximum is taken over all partitions (f^k) of f with $\|f^k\|\leqslant B$ for each k. Note that for the choice of B it follows that $\Psi(h)-\Gamma(h)\leqslant \varepsilon_0\|h\|$ for all profiles h. Then, by Lemma 1, there is an integer ν_0 sufficiently large so that, for all $\nu\geqslant \nu_0$,

$$\begin{aligned}&\left|\frac{\Psi(f^\nu)}{\|f^\nu\|}-\frac{\Psi(g^\nu)}{\|g^\nu\|}\right|\\&\quad\leqslant\left|\frac{\Psi(f^\nu)}{\|f^\nu\|}-\frac{\Gamma(f^\nu)}{\|f^\nu\|}\right|+\left|\frac{\Gamma(f^\nu)}{\|f^\nu\|}-\frac{\Gamma(g^\nu)}{\|g^\nu\|}\right|+\left|\frac{\Gamma(g^\nu)}{\|g^\nu\|}-\frac{\Psi(g^\nu)}{\|g^\nu\|}\right|\\&\quad\leqslant 3\varepsilon_0,\end{aligned}$$

a contradiction to (7.12). *Q.E.D.*

It is convenient to prove Theorem 4 next.

PROOF OF THEOREM 4: Let (T,Ψ) be a pregame with bounded per capita payoffs and suppose that the conclusion of the Theorem is false. Then there are real numbers ρ_0 and ε_0 and a sequence of profiles $\{f^\nu\}$ such that

$$\|f^\nu\|\to\infty \quad \text{as} \quad \nu\to\infty,$$
$$f^\nu\in P(\rho) \quad \text{for each } \nu, \text{ and}$$

for each f^ν, for every partition $(f^{\nu k})$ of f^ν with $\|f^{\nu k}\|<\nu$ for each k, it holds that

$$\Psi(f^\nu)-\sum_k \Psi(f^{\nu k})>2\varepsilon_0\|f^\nu\|. \tag{7.13}$$

Without loss of generality we can assume that for some $Q\leqslant T$, $f^\nu\in R^Q_{++}\times\{0\}^{T\setminus Q}$ for each ν. By passing to a subsequence if necessary, we can suppose that the sequence $\{(1/\|f^\nu\|)f^\nu\}$ converges, say to $f\in R^Q_{++}\times\{0\}^{T\setminus Q}$. Again passing to a subsequence if necessary, from per capita boundedness we can suppose that the sequence $\{\Psi(f^\nu)/\|f^\nu\|\}$ converges, say to the real number L. Since the sequence $\{\Psi(f^\nu)/\|f^\nu\|\}$ converges, there is an integer ν_0 sufficiently large so that

$$\frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\leqslant L+\varepsilon_0$$

and for all $\nu\geqslant\nu_0$ it holds that

$$\left|\frac{\Psi(f^\nu)}{\|f^\nu\|}-\frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\right|<\varepsilon_0.$$

Let ν_1 be sufficiently large so that for each $\nu \geqslant \nu_1$, for some integer r_ν and profile m^ν it holds that

$$f^\nu = r_\nu f^{\nu_0} + m^\nu, \quad \text{and}$$

$$\frac{\|m^\nu\|}{\|f^\nu\|}(L+\varepsilon_0) \leqslant \varepsilon_0;$$

this is possible since each term f^ν is in $P(\rho)$. (A proof of this fact is provided in Wooders and Zame (1987).) From the above inequalities, for all ν sufficiently large we obtain the estimate:

$$\begin{aligned}
&\left|\frac{\Psi(f^\nu)}{\|f^\nu\|} - r_\nu \frac{\Psi(f^{\nu_0})}{\|f^\nu\|}\right| \\
&\leqslant \left|\frac{\Psi(f^\nu)}{\|f^\nu\|} - \frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\right| + \left|\frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|} - r_\nu \frac{\Psi(f^{\nu_0})}{\|f^\nu\|}\right| \\
&\leqslant \varepsilon_0 + \frac{\Psi(f^{\nu_0})}{\|f^{\nu_0}\|}\frac{\|m^\nu\|}{\|f^\nu\|} \\
&\leqslant \varepsilon_0 + (L+\varepsilon_0)\frac{\|m^\nu\|}{\|f^\nu\|} \leqslant 2\varepsilon_0.
\end{aligned}$$

This yields a contradiction to (7.13) since for all ν sufficiently large it follows that

$$\Psi(f^\nu) - r_\nu \sum_k \Psi(f^{\nu_0}) - \sum_t m_t^\nu \Psi(\chi^t) \leqslant 2\varepsilon_0 \|f^\nu\|.$$

We leave the other direction to the reader. *Q.E.D.*

PROOF OF THEOREM 1: Let (I, Λ) be a market pregame derived from a premarket $(\mathbf{R}_+^M, A)$ with I types of participants and where all participants have the same 1-homogeneous and continuous payoff function, denoted by u. Suppose (I, Λ) does not satisfy small group effectiveness. Then there is a positive real number ε_0 and a sequence of profiles $\{f^\nu\}$ with the property that for any partition $(f^{\nu k})$ of f^ν where, for each k, $\|f^{\nu k}\| \leqslant \nu$ it holds that

$$(7.14) \quad \Lambda(f^\nu) - \sum_k \Lambda(f^{\nu k}) > 3\varepsilon_0 \|f^\nu\|.$$

Suppose, without loss of generality, that there is a vector f in $\mathbf{R}_+^I$ such that $(1/\|f^\nu\|)f^\nu$ converges to f. Define the sequences $\{h^\nu\}$ and $\{l^\nu\}$ by equations (7.7), (7.8), and (7.9). Since the sequence $\{h^\nu\}$ has the property that the percentage of players of each type (that appears in the profiles) is bounded away from zero, from Theorem 4 there is an integer B such that for each ν, for some partition $(h^{\nu k})$ of h^ν with $\|h^{\nu k}\| \leqslant B$ for each k,

$$(7.15) \quad \Lambda(h^\nu) - \sum_k \Lambda(h^{\nu k}) \leqslant \varepsilon_0 \|h^\nu\|.$$

Note that since u is concave, for any profile g it holds that $\Lambda(f) = u(\Sigma_i g_i\omega^i)$. From Rockafellar (1972, Theorem 10.4), u is Lipschitzian on the simplex. Since both u and Λ are 1-homogeneous, there is a constant A such that for each integer ν

$$(7.16) \qquad \left|\frac{\lambda(f^\nu)}{\|f^\nu\|} - \frac{\Lambda(h^\nu)}{\|h^\nu\|}\right| \leqslant A\left\|\frac{1}{\|f^\nu\|}f^\nu - \frac{1}{\|h^\nu\|}h^\nu\right\|.$$

Select an integer ν_0 sufficiently large so that

$$A\left\|\frac{1}{\|f^\nu\|}f^\nu - \frac{1}{\|h^\nu\|}h^\nu\right\| < \varepsilon_0.$$

From superadditivity, (7.15) and (7.16), and the fact that $(\|h^\nu\|/\|f^\nu\|) \to 1$ as η becomes large, it follows that for all sufficiently large ν

$$\left|\frac{\Lambda(f^\nu)}{\|f^\nu\|} - \frac{\Lambda(h^\nu)}{\|f^\nu\|}\right| \leqslant 2\varepsilon_0.$$

It now follows that for all sufficiently large ν,

$$\begin{aligned}\Lambda(f^\nu) &- \sum_k \Lambda(h^{\nu k}) - \sum_i l_i^\nu \Lambda(\chi_i)\\ &\leqslant |\Lambda(f^\nu) - \Lambda(h^\nu)| + \left|\Lambda(h^\nu) - \sum_k \Lambda(h^{\nu k})\right|\\ &\leqslant 3\varepsilon_0\|f^\nu\|,\end{aligned}$$

which is a contradiction. *Q.E.D.*

Proof of Theorem 2: Let (T, Ψ) be a pregame. From Lemma 2 the function u, given by (6.1), is well-defined and continuous. Since it is 1-homogeneous and superadditive, u is concave.

Now consider the premarket $(\mathbf{R}_+^T, A)$ where $|A| = T$ and

$$a^i = (\chi_t, u)$$

for each $t = 1, \ldots, T$. The pair $(\mathbf{R}_+^T, A)$ is a continuous, socially homogeneous premarket with payoff-constant returns. From the constraints of u it follows that the pregame (T, Ψ) is asymptotically equivalent to the pregame derived from $(\mathbf{R}_+^T, A)$; we leave the details to the reader. This proves one part of the Theorem.

Next, let $(\mathbf{R}_+^M, A)$ be a socially homogeneous premarket with derived game (T,Λ) and with the property that (T, Λ) and (T, Ψ) are asymptotically equivalent. Let u denote the payoff function of the participants in the premarket. From Theorem 1 (T, Λ) has effective small groups. Thus, given $\varepsilon > 0$ there is an integer $\eta_0(\varepsilon)$ such that for each profile f there is a partition of f, say $(f^k: k = 1, \ldots, K)$, satisfying

$$(7.17) \qquad \|f^k k\| \leqslant \eta_0(\varepsilon) \quad \text{and}$$

$$(7.18) \qquad \Lambda(f) - \sum_k \Lambda(f^k) < \varepsilon\|f\|.$$

From asymptotic equivalence there is an integer $\eta_1(\varepsilon) \geqslant \eta_0(\varepsilon)$ such that for each profile f with

(7.19) $\quad \|f\| \geqslant \eta_1(\varepsilon)$

it holds that

(7.20) $\quad |\Psi(f) - \Lambda(f)| \leqslant \varepsilon\|f\|.$

The following Lemma leads to the conclusion of the proof.

LEMMA 3: *Let f be a profile and let $(f^k\colon k = 1,\dots,K)$ be a partition of f satisfying* (7.17) *and* (7.18). *Then there are integers $0 = m_0 < m_1 < \dots < m_q < \dots < m_Q = K$ such that, for each $q = 0,\dots,Q-1$,*

$$\eta_1(\varepsilon) \leqslant \Big\| \sum_{k=m_q+1}^{m_{q+1}} f^k \Big\| \leqslant 4\eta_1(\varepsilon).$$

PROOF OF LEMMA 3: Suppose $\|f\| \leqslant 4\eta_1(\varepsilon)$. In this case, the partition (f) satisfies the required properties. Therefore we suppose that $\|f\| > 4\eta_1(\varepsilon)$. Let (f^k) be a partition of f satisfying (7.17) and (7.18). There exists an integer m_1 such that

$$\Big\| \sum_{k=1}^{m_1-1} f^k \Big\| \leqslant \eta_1(\varepsilon) \quad \text{and}$$

$$\Big\| \sum_{k=1}^{m_1} f^k \Big\| > \eta_1(\varepsilon).$$

It follows that

$$\left\| \sum_{k=1}^{m_1} f^k \right\| = \left\| \sum_{k=1}^{m_1-1} f^k \right\| + \|f^{m_1}\| \leqslant \eta_1(\varepsilon) + \eta_0(\varepsilon) \leqslant 2\eta_1(\varepsilon) \quad \text{and}$$

$$\left\| \sum_{k=m_1+1}^{K} f^k \right\| = \|f\| - \left\| \sum_{k=1}^{m_1} f^k \right\| \geqslant 4\eta_1(\varepsilon) - 2\eta_1(\varepsilon) = 2\eta_1(\varepsilon).$$

Case 1: If $2\eta_1(\varepsilon) \leqslant \|\sum_{k=m_1+1}^{K} f^k\| \leqslant 4\eta_1(\varepsilon)$ let $g^1 = \sum_{k=1}^{m_1}$ and let $g^2 = \sum_{k=m_1+1}^{K} f^k$.

Case 2: If $\|\sum_{k=m_1+1}^{K} f^k\| > 4\eta_1(\varepsilon)$, then there is an integer m_2 such that

$$\nu_1(\varepsilon) \leqslant \left\| \sum_{k=m_2+1}^{m_2} f^k \right\| \leqslant 2\eta_1(\varepsilon) \quad \text{and}$$

$$\left\| \sum_{k=m_2+1}^{K} f^k \right\| > 2\eta_1(\varepsilon).$$

In this case, define $g^1 = \sum_{k=1}^{m_1} f^k$, and $g^2 = \sum_{k=m_1+1}^{m_2} f^k$.

Next consider $\Sigma_{k=m_2+1}^{K} f^k$. Depending on whether

$$\left\| \sum_{k=m_2+1}^{K} f^k \right\| \leqslant 4\eta_1(\varepsilon) \quad \text{or} \quad \left\| \sum_{k=m_2+1}^{K} f^k \right\| > 4\eta_1(\varepsilon)$$

proceed as in Case 1 or 2 to determine m_3 and define $g^3 = \Sigma_{k=m_2+1}^{m_3} f^k$, satisfying the required conditions.

We can argue repeatedly as above to determine integers $m_1, \dots, m_Q$ and profiles $g^1, \dots, g^Q$ such that

$$g^1 = \sum_{k=1}^{m_1} f^k,$$

$$g^2 = \sum_{k=m_1+1}^{m_2} f^k, \dots, \quad \text{and}$$

$$g^Q = \sum_{k=m_{Q-1}+1}^{K} f^k,$$

where $\eta_1(\varepsilon) \leqslant \|g^q\| \leqslant 4\eta_1(\varepsilon)$ for all $q = 1, \dots, m_Q$. This completes the proof of the Lemma. *Q.E.D.*

To complete the proof of the Theorem, from superadditivity, (7.17) and (7.18), and Lemma 3 it now follows that

$$\begin{aligned}
&\Psi(f) - \sum_{q}^{Q} \Psi(g^q) \\
&\quad \leqslant |\Psi(f) - \Lambda(f)| + \left| \Lambda(f) - \sum_{q=1}^{Q} \Lambda(g^q) \right| \\
&\qquad + \left| \sum_{q=1}^{Q} \Lambda(g^q) - \sum_{q=1}^{Q} \Psi(g^q) \right| \\
&\quad \leqslant 3\varepsilon \|f\|,
\end{aligned}$$

where $\{g^q\}$ is defined as in the proof of Lemma 3. Thus, given any $\varepsilon > 0$ the integer $4\eta_1(\varepsilon)$ satisfies the properties required in the definition of small group effectiveness applied to the pregame (T, Ψ). This completes the proof of the Theorem. *Q.E.D.*

Theorem 5 is a standard result. With the appropriate construction of a premarket from a pregame satisfying per capita boundedness, the proof of Theorem 6 is essentially the same as the proof of Theorem 2. We note only that for any x in $\mathbf{R}_{++}^T$ the definition of the payoff function for the premarket approximating the pregame can be taken as that given by (6.1) with the approximating profiles taken to have the same support as the commodity bundle.

Theorem 8 was proved during the course of the proof of Theorem 2 so we have reached the conclusion of this section.

8. DISCUSSION

Economic models with effective small groups may include ones with multiple marketplaces, firms, communities, or clubs. Small group effectiveness ensures that when there is in total a large number of participants, then there are, at least potentially, "many" groups. The intuition underlying our results is that in economies with effective small groups, competition within groups for shares of the surplus generated by the group, and competition between groups for participants, lead to a competitive outcome. This intuition emerges especially from the study of economies with collectively produced and/or consumed goods, such as ones with local public goods (see, for example, Tiebout (1956) and Wooders (1980)). It follows from the results of the current paper that one set of commodities for which an economy is approximately a market is the set of participants themselves. Intuitively, participants sell their participation in groups in return for a share of the surplus generated by the group.[15]

Our concepts and results can be applied in a variety of contexts, for example, in investigations of the "Coase Theorem" (Coase (1960)). One such investigation of the effects of property rights assignments is carried out in Wooders (1992c). Provided that assignments of property rights are bounded, small group effectiveness of pregames (called "technologies" in this application, as in Wooders and Zame (1987)) ensures convergence of approximate cores to competitive prices for attributes of players and to approximate attribute core payoffs. Other applications include ones to economies with collective production and to economies with local public goods or shared goods more generally; some such applications are reviewed in Wooders (1992b). Another application may be to financial markets with unbounded short sales.

The approach of this paper is quite distinct from established theories of competitive markets. Three major theories are Cournot's noncooperative equilibrium theory, Edgeworth's contract theory, and Clark's marginal productivity theory, revived in Ostroy's no-surplus theory.[16] The equivalence result of this paper suggests that small group effectiveness may be a powerful condition for the study of these theories; the theories have been applied to markets, and asymptotically economies with effective small groups are markets. Some results

[15] These ideas have been a reoccurring theme in the research leading to this paper. Ostroy (1984) also stresses prices for participants.

[16] See Cournot (1938), and more recently, Shubik (1973), Hart (1979), and Novshek and Sonnenschein (1978) for the Cournotian theory, Anderson (1992) for a survey of research on Edgeworth's contract theory and the equivalence of cooperation and equilibrium outcomes, and Ostroy (1980, 1984) for the marginal productivity theory.

in these directions include Hammond, Kaneko, and Wooders (1989) on the equivalence of the core and the competitive outcomes in large economies with widespread externalities and effective small groups (finite coalitions in the continuum).

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