Duration-Based Volatility Estimation
A Dual Approach to RV

Torben G. Andersen, Northwestern University
Dobrislav Dobrev, Federal Reserve Board of Governors
Ernst Schaumburg, Northwestern University

CHICAGO-ARGONNE
INSTITUTE ON COMPUTATIONAL ECONOMICS
University of Chicago, August 8th, 2008
The prevailing approach to high frequency volatility estimation is to measure the **price change (return) per time unit**

We put forward the **dual approach** of measuring the **time duration (passage time) per unit price change**

Our analysis fills an existing gap in the IV estimation literature by:

1. Developing a broad class of duration-based IV estimators
2. Identifying situations in which the duration-based approach is advantageous
3. Shedding new light on the microstructure properties of real data
Literature Review

Magnitude-based approach to volatility estimation:
1. Discrete time (parametric): rich ARCH literature
2. Continuous time (non-parametric): rich RV literature

Duration-based approach to volatility estimation:
1. Discrete time (parametric): rich ACD literature
2. Continuous time (non-parametric): lack of “DV” literature

*Cho and Frees (JF ’88) consider non-parametric duration-based estimation under constant volatility but their procedure is not suitable for IV estimation*
Our Contribution

• Develop duration-based analogues to RV, range-based RV (RRV), and other power/multipower variations

• Derive an asymptotic theory for our duration-based estimators showing consistency for IV and promising asymptotic efficiency

• Demonstrate excellent finite sample efficiency and robustness to both (finite activity) jumps and microstructure noise

• Document superior performance in comparison to subsampled BV both on simulated and real stock data
Main Idea

- Inherent duality between the **increment** \( h \) and corresponding **passage time** \( dt \) for a Brownian motion:

\[
\mathbb{E}[h^2 | dt] \sim \sigma^2 dt \quad \quad \quad \quad \mathbb{E}[dt | h] \sim \frac{h^2}{\sigma^2}
\]

- Passage time moment subject to **Jensen effect**!
  - Resolution: Use the reciprocal passage time

\[
\mathbb{E}\left[ \frac{1}{dt} | h \right] \sim \frac{\sigma^2}{h^2} \quad \Rightarrow \quad \hat{\sigma}_h^2 = \text{const} \times \frac{h^2}{dt}
\]

- Passage time moment subject to **Censoring effect**!
  - Resolution: Exploit the time reversibility of the Brownian motion

- Passage time moment subject to **Discretization effect**!
  - Resolution: Apply discretization error theory for Brownian maxima
Consider a fixed time grid $0 = t_0 < t_1 < \ldots < t_N = 1$ consisting of $N$ intervals with mesh size $\Delta_i = t_{i+1} - t_i$.

From a sequence of unbiased local variance estimates $\hat{\sigma}^2_h(t_i)$ we can construct an IV estimate:

Define $DV$ as a duration-based counterpart to RV based on a sequence of reciprocal passage times instead of squared returns:

$$\hat{DV}_{N,h} = \sum_{i=0}^{N-1} \hat{\sigma}^2_h(t_i) \Delta_i$$
Brownian Passage Times - A Multitude of Definitions

- **Forward passage times for threshold** \( h \):
  \[
  \tau^+_h(t) = \left\{ \begin{array}{ll}
  \inf_{\theta > 0} \{ W_{t+\theta} - W_t = h \} & \text{(first hitting time)} \\
  \inf_{\theta > 0} \{ |W_{t+\theta} - W_t| = h \} & \text{(first exit time)} \\
  \inf_{\theta > 0} \{ \sup_{[t,t+\theta]} W_s - \inf_{[t,t+\theta]} W_s = h \} & \text{(first range time)}
  \end{array} \right.
  \]

- **Backward passage times for threshold** \( h \):
  \[
  \tau^-_h(t) = \left\{ \begin{array}{ll}
  \inf_{\theta > 0} \{ W_{t-\theta} - W_t = h \} & \text{(first hitting time)} \\
  \inf_{\theta > 0} \{ |W_{t-\theta} - W_t| = h \} & \text{(first exit time)} \\
  \inf_{\theta > 0} \{ \sup_{[t-\theta,t]} W_s - \inf_{[t-\theta,t]} W_s = h \} & \text{(first range time)}
  \end{array} \right.
  \]
Brownian Passage Times - A Multitude of Definitions

- First Range Time
- First Hitting Time
- First Exit Time
A Multitude of DV Estimators

\[ \hat{DV}_{N,h} = \sum_{i=0}^{N-1} \hat{\sigma}_h^2(t_i) \Delta_i \]

- Local estimators of \( \sigma^2 \) given by the scaled reciprocal passage times:
  \[ \hat{\sigma}_h^2(t_i) = \mu_1^{-1} \frac{h^2}{\tau_h} \]

- **First exit time DV:** based on reciprocal first exit times
- **First range time DV:** based on reciprocal first range times

- More generally, local estimators of \( \sigma^p \) are given by the \( p/2 \) power of the reciprocal passage times:
  \[ \hat{\sigma}_h^p(t_i) = \mu_{p/2}^{-1} \frac{h^p}{\tau_h^{p/2}} \]

- Can define natural DV analogues to power and multipower variations
Duality

Theorem (Duality for magnitude and passage time functionals)

Define the following standard Brownian functionals

\[
H_t = \sup_{\theta \in [0;t]} B_\theta, \quad M_t = \sup_{\theta \in [0;t]} |B_\theta|, \quad R_t = \sup_{\theta \in [0;t]} B_\theta - \inf_{\theta \in [0;t]} B_\theta
\]

and let (for \( h > 0 \))

\[
\tau_{HT}^h = \inf \{ t \mid H_t = h \}, \quad \tau_{ET}^h = \inf \{ t \mid M_t = h \}, \quad \tau_{RT}^h = \inf \{ t \mid R_t = h \}
\]

be the first range time, first exit time, and first hitting time respectively. Then we have the following identities in distribution:

\[
H_1 \overset{\mathcal{D}}{=} \frac{1}{(\tau_{1HT}^h)^{1/2}}, \quad M_1 \overset{\mathcal{D}}{=} \frac{1}{(\tau_{1ET}^h)^{1/2}}, \quad R_1 \overset{\mathcal{D}}{=} \frac{1}{(\tau_{1RT}^h)^{1/2}}
\]
We establish the following main asymptotic result:

\[
\sqrt{N} \left( \hat{DV}_{N,h} - IV \right) \sim \text{Mixed Normal} \left( 0, \nu \int_0^1 \sigma_u^4 \, du \right),
\]

where

\[
\nu \approx \begin{cases} 
0.7681 & \text{(first exit time)} \\
0.4073 & \text{(first range time)} \\
2.0000 & \text{(first hitting time)}
\end{cases}
\]

is the variance factor of the individual passage time estimators at each grid point.

Underlying assumptions:
- No leverage
- Lipschitz continuity of the volatility process
- Mesh size \( \Delta = O \left( N^{-1} \right) \), threshold \( h = o \left( N^{-1/2} \right) \)
In practice, the observation record is discrete and we only observe the value of the process at \( N \) grid points but not in between.

For a feasible version of DV consider coarser sub-grid \( \{ t_{i_1}, \ldots, t_{i_K} \} \) where \( K = o(N), \ K \to \infty \):

\[
\sqrt{K} \left( \hat{DV}_{K,h} - IV \right) \sim \text{Mixed Normal} \left( 0, \nu \int_0^1 \sigma_u^4 \, du \right)
\]

Convergence rate is now the slower \( K^{-1/2} \).

Efficiency loss can be mitigated by averaging the estimator over all possible sub-grids of mesh size \( \Delta = K^{-1} \).

The outcome is feasible DV akin to subsampled RV!
Main Challenges to IV Estimation

Microstructure noise

Jumps
An intuitive advantage of the passage time approach is that the threshold $h$ can be chosen large enough to achieve noise-robustness.

To formalize this intuition we adopt an AR(1) noise structure:

$$\tilde{p}_i = p_i + u_i$$
$$u_i = \rho u_{i-1} + \varepsilon_i,$$

where $\varepsilon_i \sim N(0, (1 - \rho^2)\omega^2)$, so that $\mathbb{E}[u_i] = 0$, $\mathbb{V}[u_i] = \omega^2$.

For $\rho = 0$ we obtain a Gaussian i.i.d. specification representative for transaction prices.

For $\rho \gg 0$ we obtain a persistent autoregressive specification representative for quotes.
Robustness of DV to Microstructure Noise Cont’d

- Given a passage time transition on noisy data, it is possible to infer the expected magnitude of the latent transition.
- AR(1) noise leads to an **upward bias** of our reciprocal passage time estimators.
- We plot the upward bias factor as a function of the noise persistence \( \rho \) for two different noise-to-signal ratios:
  1. \( \lambda = 0.25 \) (moderate noise)
  2. \( \lambda = 1.00 \) (high noise)
Negligible bias for moderate noise ($\lambda = 0.25$) at all persistence levels $\rho$. 

Andersen, Dobrev & Schaumburg

Duration-Based Volatility Estimation
Negligible bias for high noise ($\lambda = 1.0$) if sufficiently persistent ($\rho > 0.9$)
Another advantage of the passage time estimators is their inherent robustness to finite jumps.

Jumps above the threshold level $h$ are **effectively truncated**!

The lower the chosen threshold, the higher the degree of jump-robustness but ... the lower the degree of noise-robustness.

To avoid lowering the threshold too much, we propose an asymptotically equivalent “previous tick” passage time estimator.

It utilizes the lower threshold level $h^-$ corresponding to the level at one tick prior to the crossing of the target threshold $h$. 
Better jump-robustness in finite samples than the standard passage times!

Andersen, Dobrev & Schaumburg  Duration-Based Volatility Estimation
Empirical Analysis of The “Robust” DV Estimators

Real Stock Data

Simulated Stock Data
We analyze the performance of DV on the Dow Jones 30 stocks for 601 trading days in the period January 1, 2005 to May 31, 2007.

We work with mid-quotes known to have relatively low and persistent noise, so DV should be unbiased for moderate threshold levels.

DV is jump-robust, so we choose 2min subsampled BV as benchmark.

We produce DV signature plots for the mean, standard deviation, and correlation of DV with 2min subsampled BV.

Focus on first exit time DV and first range time DV for thresholds from 1 to 10 log-spreads.
BV is downward biased at higher frequencies, so 2min is close to optimal!
DV for thresholds above 4 log-spreads has **the same mean as BV**!
DV has markedly lower standard deviation than BV!
DV for thresholds above 4 log-spreads is **highly correlated with BV**!
Monte Carlo Study of DV

- Very similar results across different SV models
- Scenario of main interest: “jump” days with 25% mean jump contribution to IV and U-shape volatility pattern
- The adopted AR(1) noise structure gives rise to five distinct cases:
  1. No noise
  2. Non-persistent noise \((\rho = 0)\) at moderate level \((\lambda = 0.25)\)
  3. Persistent noise \((\rho = 0.99)\) at moderate level \((\lambda = 0.25)\)
  4. Non-persistent noise \((\rho = 0)\) at high level \((\lambda = 1.00)\)
  5. Persistent noise \((\rho = 0.99)\) at high level \((\lambda = 1.00)\)
- Compare the performance of DV vis-a-vis 2min subsampled BV by:
  (i) A relative bias measure: mean of \(\hat{IV}/IV\)
  (ii) A relative MSE measure: mean of \(195(\hat{IV} − IV)^2/IQ\)
Monte Carlo Study of DV 1/5: No Noise

Model SV2A-UJ with U-Shaped Pattern and 25% Mean Jump Contribution to IV
Avg Sample Freq 3sec without Microstructure Noise

Robust Exit Time DV
Robust Range Time DV
2min Subsampled BV

Relative Bias vs. Threshold Factor (x Average Log Spread)

Relative MSE vs. Threshold Factor (x Average Log Spread)

Andersen, Dobrev & Schaumburg
Duration-Based Volatility Estimation
Monte Carlo Study 2/5: Moderate Nonpersistent Noise

Model SV2A-UJ with U-Shaped Pattern and 25% Mean Jump Contribution to IV
Avg Sample Freq 3sec with Non-Persistent Noise at Moderate Level

Robust Exit Time DV
Robust Range Time DV
2min Subsampled BV

Relative Bias
Threshold Factor (x Average Log Spread)

Relative MSE
Threshold Factor (x Average Log Spread)
Monte Carlo Study of DV 3/5: Moderate Persistent Noise

Model SV2A-UJ with U-Shaped Pattern and 25% Mean Jump Contribution to IV
Avg Sample Freq 3sec with Persistent Noise at Moderate Level

Robust Exit Time DV
Robust Range Time DV
2min Subsampled BV

Relative Bias

Threshold Factor (x Average Log Spread)

Relative MSE

Andersen, Dobrev & Schaumburg
Duration-Based Volatility Estimation
Monte Carlo Study of DV 4/5: High Nonpersistent Noise

Model SV2A-UJ with U-Shaped Pattern and 25% Mean Jump Contribution to IV
Avg Sample Freq 3sec with Non-Persistent Noise at High Level

Robust Exit Time DV
Robust Range Time DV
2min Subsampled BV

Relative Bias

Threshold Factor (x Average Log Spread)

Relative MSE

Andersen, Dobrev & Schaumburg
Duration-Based Volatility Estimation
Monte Carlo Study of DV 5/5: High Persistent Noise

Model SV2A-UJ with U-Shaped Pattern and 25% Mean Jump Contribution to IV
Avg Sample Freq 3sec with Persistent Noise at High Level

Robust Exit Time DV
Robust Range Time DV
2min Subsampled BV

Relative Bias
Threshold Factor (x Average Log Spread)

Relative MSE
Threshold Factor (x Average Log Spread)

Andersen, Dobrev & Schaumburg
Duration-Based Volatility Estimation
Summary and Conclusions

- Novel dual approach to realized return variation measurement based on reciprocal passage times
- Duration-based counterparts to RV, range-based RV (RRV), and other power/multipower variations
- Asymptotic theory showing consistency for IV and promising asymptotic efficiency
- Natural robustness to both jumps and microstructure noise
- Promising finite sample efficiency in comparison to subsampled BV both on simulated and real stock data
- Exciting “to do” list!