Bond Portfolios and Two-Fund Separation in the Lucas Asset-Pricing Model

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Modern Portfolio Theory

Asset allocation in stochastic environments
Optimal investment in stocks, bond, and cash

Partial equilibrium analysis:
Exogenously specified stochastic processes of returns and interest rate

Continuous-time literature based on Merton (1973)
Many recent examples, e.g. Brennan and Xia (2000, 2002), Wachter (2003)

Discrete-time factor models
Campbell and Viceira (2001, 2002)
Motivation: This Paper

Popular models are partial equilibrium and not GE models

Few underlying factors, markets are complete with very few assets

Analysis of complex bond portfolios impossible

Our paper: Follow very different approach

Examine investors’ portfolios in a dynamic GE model

Lucas asset pricing model with heterogeneous agents and many states of nature

Dynamically complete security markets

Market completeness through presence of many bonds
Summary: One Bond

HARA utility, linear sharing rules

Two-fund separation hinges on maturity of the bond

Consol: two-fund separation

One-period bond: typically no two-fund separation

Consol: riskless asset in an infinite-horizon dynamic model
  safe consumption stream over the infinite horizon
  uncertain capital value does not affect portfolios

One-period bond: risky asset in an infinite-horizon dynamic model
  time-varying interest rates, reinvestment risk
Summary: Many Bonds

Dynamically complete security markets with several zero-coupon bonds
Bonds have maturities of 1, 2, \ldots, K periods

Stock portfolios typically do not exhibit two-fund separation
Bond portfolios involve unrealistically large trading volume of long-term bonds

As the number of states and bonds increases:
Stock portfolios approach two-fund separation
Bond portfolios show laddering structure for short maturities

Two-fund separation and bond ladders are approximately optimal
Introduction of redundant bonds is welfare-improving
Overview

- Dynamic GE Model
- HARA Utility Functions and Linear Sharing Rules
- Separation Results for the GE Model with a single bond
- Families of Finite-Maturity Bonds
- Bond Ladders
General Equilibrium Model

Lucas asset pricing model with heterogeneous agents
Dynamically complete asset markets

Markov process of exogenous dividend states, \( y \in \mathcal{Y} = \{1, \ldots, Y\} \)
Transition matrix \( \Pi \gg 0 \)

Finite number of types of infinitely-lived agents, \( h \in \mathcal{H} = \{1, \ldots, H\} \)
Single perishable consumption good (produced by firms)

Agents have no individual endowments but hold an initial portfolio of firms’ stock
Firms distribute output through dividends (“Lucas trees”)
Securities

Infinitely-lived stocks with dividends $d^j : \mathcal{Y} \to \mathbb{R}_{++}$ for $j = 1, \ldots, J$

Stocks are in unit net supply

Each agent has initial holding of stocks

Initial model: two types of bonds

Consol with safe payoff $d^c_y = 1$ for all $y \in \mathcal{Y}$

One-period bond with safe payoff next period

Bonds are in zero net supply

Agents hold no initial positions
Utility Function

Time-separable utilities

\[ U_h(c) = E \left\{ \sum_{t=0}^{\infty} \beta^t u_h(c_t) \right\} \]

Consumption process \( c = (c_0, c_1, \ldots) \)

\( u_h : \mathbb{R}_{++} \rightarrow \mathbb{R} \) strictly monotone, \( C^2 \), and strictly concave

Identical discount factor \( \beta \in (0, 1) \) for all agents
Equilibrium

Complete markets: Pareto efficient consumption allocations
Consumption only depends on current dividend state, is independent of history and any other state variables
Consumption “process” is represented by a vector of $Y$ numbers
Negishi approach determines allocations; nonlinear system of equations

Portfolios are constant for $Y$ independent dividend vectors
State-independent portfolio of stocks $\psi^h$, and consol $\theta^h_c$ or bond $\theta^h_1$
Budget equations determine constant portfolios; linear system of equations
Classical Two-Fund Separation

Tobin (1958), Markowitz (1959)

There is a riskless asset and the agent has HARA utility
Monetary separation: The relative allocation of wealth across risky assets is invariant to wealth and risk attitude
General Equilibrium

Market-clearing in general equilibrium model

Two-fund separation \( \iff \psi_j^h = \psi_{j'}^h \quad \forall \, j, j' \)

Rubinstein (1974): Equi-cautious HARA utility leads to linear sharing rules for all agents in static GE

Generalizes to our dynamic model: \( c_y^h = m_y^h e_y + b_y^h \quad \forall \, h, \forall \, y \)

Social endowment \( e_y = \sum_{j=1}^{J} d_y^j \)
Consol vs. One-period Bond

Consol

\[ m^h e_y + b^h = c^h_y = \sum_{j=1}^{J} \psi_j^h d^j_y + \theta_c^h \cdot 1 \]

Two-fund separation holds, \( \theta_c^h = b^h \) and \( \psi_j^h = m^h \) \( \forall j \)

One-period bond

\[ m^h e_y + b^h = c^h_y = \sum_{j=1}^{J} \psi_j^h d^j_y + \theta_1^h \cdot (1 - q_y^1) \]

Generically no two-fund separation when \( b^h \neq 0 \)

Deviations from two-fund separation are quantitatively significant
Many Finite-Maturity Bonds

Infinitely-lived stocks with dividends $d^j : \mathcal{Y} \rightarrow \mathbb{R}_{++}$ for $j = 1, \ldots, J$

$K$ zero-coupon bonds of maturities $1, 2, \ldots, K$ in zero net supply

Agent $h$’s bond portfolio, $\theta^h_1, \theta^h_2, \ldots, \theta^h_K$

Agent $h$’s budget constraint (in stationary equilibrium)

$$c^h_y = \sum_{j=1}^{J} \psi^h_j d^j_y + \theta^h_1 (1 - q^1_y) + \sum_{k=2}^{K} \theta^h_k (q^{k-1}_y - q^k_y)$$
Two-Fund Separation with IID Dividends

Any number $J$ of stocks, two bonds with maturities $k = 1, 2$

IID beliefs over next period’s dividend states

Prices of the two bonds are perfectly correlated, $q^2 = \beta q^1$

Portfolio

$$\theta^h_1 = b^h, \quad \theta^h_2 = \frac{b^h}{(1 - \beta)},$$

implements holding $b^h$ of consol and so creates safe consumption stream of size $b^h$

Two-fund separation for stock portfolio $\psi^h_j = m^h \forall j = 1, \ldots, J$
Spanning the Consol

Finite-maturity bonds span consol \( \implies \) two-fund separation

\[
c^h_y = m^h c^h_y + b^h = \sum_{j=1}^{J} \psi^h_j d^j_y + \theta^h_1 (1 - q^1_y) + \sum_{k=2}^{K} \theta^h_k (q^k_{y} - q^{k-1}_{y})
\]

Spanning means relationship between bond price vectors \( q^1, q^2, \ldots, q^K \)

Sufficient conditions for spanning

Key ingredient is Markov transition matrix \( \Pi \) of exogenous shocks
Examples with Many Bonds

\( J \) independent stocks with independent high and low dividends

<table>
<thead>
<tr>
<th>stock</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tr>
<td>low</td>
<td>0.98</td>
<td>0.77</td>
<td>0.95</td>
<td>0.8</td>
<td>0.91</td>
<td>0.86</td>
<td>0.9</td>
</tr>
<tr>
<td>pers.</td>
<td>0.55</td>
<td>0.81</td>
<td>0.61</td>
<td>0.74</td>
<td>0.66</td>
<td>0.7</td>
<td>0.68</td>
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</table>

Model with \( J \) independent stocks: \( Y = 2^J \) states

With different persistence, \( K = Y - J \) bonds

\( H = 2 \) agents with power utilities, \( \frac{1}{1-\gamma} (c - A^h)^{1-\gamma} \), \( \beta = 0.95 \)
Sharing rule for agent 1, $c^1_y = 0.3 \cdot e_y + 0.2$

<table>
<thead>
<tr>
<th>$(J, K)$ =</th>
<th>(3, 5)</th>
<th>(4, 12)</th>
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<th>(6, 58)</th>
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<td>$\psi_1$</td>
<td>0.30</td>
<td>$\psi_1$</td>
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<td>0.30</td>
<td>$\psi_2$</td>
<td>0.30</td>
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<tr>
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<td>0.30</td>
<td>0.30</td>
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</tr>
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<td>0.30</td>
<td>0.30</td>
<td>$\psi_5$</td>
<td>0.30</td>
<td>$\psi_5$</td>
<td>0.30</td>
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<td>0.20</td>
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<td>$\theta_8$</td>
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<td>$\theta_3$</td>
<td>0.20</td>
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<td>$\theta_4$</td>
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<td>0.20</td>
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<td>0.20</td>
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<td>$\theta_7$</td>
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<td>-5.2</td>
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<td>$\theta_8$</td>
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<td>-423</td>
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<td>-4627</td>
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<tr>
<td>$\theta_{12}$</td>
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<td>146</td>
<td>$\theta_{58}$</td>
<td>998</td>
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Deviations from Two-Fund Separation

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<tr>
<th>$J$</th>
<th>$K$</th>
<th>$\Delta^S$</th>
<th>$\Delta^1$</th>
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<th>$\Delta^3$</th>
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<th>$\Delta^5$</th>
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<td>4</td>
<td>12</td>
<td>4.5 (-9)</td>
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<td>2.0 (-6)</td>
<td>1.1 (-4)</td>
<td>3.7 (-3)</td>
</tr>
<tr>
<td>5</td>
<td>27</td>
<td>3.5 (-33)</td>
<td>6.3 (-34)</td>
<td>8.3 (-31)</td>
<td>8.3 (-28)</td>
<td>4.6 (-25)</td>
<td>1.6 (-22)</td>
</tr>
<tr>
<td>6</td>
<td>58</td>
<td>9.6 (-88)</td>
<td>4.2 (-85)</td>
<td>3.1 (-81)</td>
<td>1.1 (-77)</td>
<td>2.1 (-74)</td>
<td>3.0 (-71)</td>
</tr>
<tr>
<td>7</td>
<td>121</td>
<td>2.0 (-222)</td>
<td>4.9 (-214)</td>
<td>1.8 (-209)</td>
<td>3.0 (-205)</td>
<td>3.2 (-201)</td>
<td>2.4 (-197)</td>
</tr>
</tbody>
</table>

As $K$ increases,

stock portfolios converge to holdings satisfying two-fund separation

holdings of bonds with short maturity are approximately $b^h$ for agent $h$
More Deviations

<table>
<thead>
<tr>
<th>$k$</th>
<th>$(5, 27)$</th>
<th>$(6, 58)$</th>
<th>$(7, 121)$</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>$3.5 \ (-20)$</td>
<td>$3.0 \ (-68)$</td>
<td>$1.4 \ (-193)$</td>
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<td>7</td>
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<td>$3.0 \ (-12)$</td>
<td>$2.9 \ (-57)$</td>
<td>$2.0 \ (-179)$</td>
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<tr>
<td>11</td>
<td>$1.5 \ (-10)$</td>
<td>$9.9 \ (-55)$</td>
<td>$4.5 \ (-176)$</td>
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<tr>
<td>12</td>
<td>$5.4 \ (-9)$</td>
<td>$2.9 \ (-52)$</td>
<td>$8.9 \ (-173)$</td>
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<td>$3.5 \ (-148)$</td>
</tr>
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<td>25</td>
<td>$555.6$</td>
<td>$1.1 \ (-25)$</td>
<td>$3.9 \ (-134)$</td>
</tr>
<tr>
<td>26</td>
<td>$423.4$</td>
<td>$5.3 \ (-24)$</td>
<td>$2.0 \ (-131)$</td>
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<td>27</td>
<td>$145.8$</td>
<td>$2.4 \ (-22)$</td>
<td>$9.1 \ (-129)$</td>
</tr>
<tr>
<td>40</td>
<td>$-$</td>
<td>$3.7 \ (-5)$</td>
<td>$1.0 \ (-96)$</td>
</tr>
<tr>
<td>50</td>
<td>$-$</td>
<td>$1179.3$</td>
<td>$4.3 \ (-75)$</td>
</tr>
<tr>
<td>56</td>
<td>$-$</td>
<td>$10178$</td>
<td>$3.0 \ (-63)$</td>
</tr>
<tr>
<td>57</td>
<td>$-$</td>
<td>$4627.2$</td>
<td>$2.3 \ (-61)$</td>
</tr>
<tr>
<td>58</td>
<td>$-$</td>
<td>$998.2$</td>
<td>$1.7 \ (-59)$</td>
</tr>
</tbody>
</table>
Consider very simple portfolios

Stock portfolios must exhibit two-fund separation
Bond portfolios must have ladder structure

\[ \psi^h_j = \hat{m}^h, \forall j = 1, \ldots, J \text{ (two-fund separation)} \]
\[ \theta^h_k = \hat{b}^h, \forall k = 1, \ldots, B \text{ (bond ladder)} \]

Welfare loss of such portfolios?
Welfare Comparison of Three Portfolios

Consumption stream $c^h$ yields lifetime utility $V^h(c^h)$

Consumption equivalent $C^h$ defined by $\sum_{t=0}^{\infty} \beta^t u^h(C^h) = V^h(c^h)$

$C^{h,0} = \text{CE for consumption stream from initial portfolio}$

$C^{h,*} = \text{CE for equilibrium consumption stream}$

$C^{h,B} = \text{CE for portfolio with bond ladder } (\hat{m}^h, \hat{b}^h) \text{ of size } B$

Welfare loss from bond ladder

$$\Delta C^h = 1 - \frac{C^{h,B} - C^{h,0}}{C^{h,*} - C^{h,0}} = \frac{C^{h,*} - C^{h,B}}{C^{h,*} - C^{h,0}}$$
Equilibrium Portfolio vs. Bond Ladder

Economy with $J = 4$ independent stocks, so $Y = 2^4$ states

<table>
<thead>
<tr>
<th>stock</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>high $d$</td>
<td>1.05</td>
<td>1.08</td>
<td>1.12</td>
<td>1.15</td>
</tr>
<tr>
<td>low $d$</td>
<td>0.95</td>
<td>0.92</td>
<td>0.88</td>
<td>0.85</td>
</tr>
</tbody>
</table>

Persistence probability of both states is 0.6 for all stocks
Welfare Losses of Bond Ladders

<table>
<thead>
<tr>
<th>$B \backslash \gamma$</th>
<th>1</th>
<th>3</th>
<th>5</th>
<th>10</th>
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</thead>
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<td>1.4 (−3)</td>
<td>7.2 (−3)</td>
<td>4.8 (−2)</td>
</tr>
<tr>
<td>2</td>
<td>5.0 (−6)</td>
<td>3.0 (−3)</td>
<td>1.3 (−2)</td>
<td>7.5 (−2)</td>
</tr>
<tr>
<td>5</td>
<td>2.4 (−10)</td>
<td>3.2 (−3)</td>
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</tr>
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<td>6.3 (−13)</td>
<td>2.6 (−3)</td>
<td>1.2 (−2)</td>
<td>7.5 (−2)</td>
</tr>
<tr>
<td>30</td>
<td>≈ 0</td>
<td>7.7 (−4)</td>
<td>5.1 (−3)</td>
<td>4.7 (−2)</td>
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<tr>
<td>50</td>
<td>≈ 0</td>
<td>1.6 (−4)</td>
<td>1.4 (−3)</td>
<td>1.9 (−2)</td>
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<tr>
<td>100</td>
<td>≈ 0</td>
<td>1.2 (−6)</td>
<td>1.3 (−5)</td>
<td>2.8 (−4)</td>
</tr>
</tbody>
</table>
Summary

Portfolio analysis in Lucas asset-pricing model with many states and bonds

Equilibrium portfolios are economically unintuitive

Simple portfolios with two-fund separation and bond ladders
  are approximately optimal

Such portfolios benefit from the introduction of redundant bonds