Bayesian Analysis of A Random Coefficient Logit Model Using Aggregate Data

Peter Rossi
University of Chicago, GSB

August 2008

Joint with Renna Jiang and Puneet Manchanda

Introduction

- Often only aggregate sales/share data are available

- Berry, Levinsohn & Pakes (1995)
  - Introduce an aggregate logit model of products' market shares
    - Integrate over individual choice probabilities to obtain market shares
    - Need an aggregate error term to avoid a deterministic system after aggregation

  - Can handle endogeneity problems

  - Generalized Method of Moments (GMM)
    - Key step: invert shares to obtain mean utilities

- We conduct Bayesian inference for this model

GMM Pros and Cons

- GMM’s advantages
  - Fewer distributional assumptions
  - Easier to implement

- GMM’s disadvantages
  - Inefficiency
  - Inference for functions of model parameters(e.g., price elasticity, price-cost margin) of parameter estimates is difficult or computationally intensive
  - Numerical problems | multiple local maxima etc …

Bayes Pros and Cons

- Bayes advantages
  - Could be more efficient
  - Stochastic search can handle irregular criterion functions
  - Finite sample inference w/o resort to asymptotic approximations for all functions of model parms

- Bayes disadvantages
  - Requires one additional distributional assumption (so what?)
  - Derivation of the likelihood
  - Reliable MCMC algorithm

Other Bayesian approaches

- Chen and Yang (2003)
  - No aggregate demand shocks

- Musalem, Bradlow & Raju (2006)
  - Apply to situations where there is a fixed, known, and relatively small set of consumers over which aggregate demand is formed.

- Romeo (2007)
  - Use GMM criterion as the basis of a pseudo-likelihood

Model

- Consumer i, product j $(0,1, \ldots , J)$, time t

$$U_{ijt} = X_{jt}\theta_i + \eta_{jt} + \varepsilon_{ijt}$$

where X_{jt} is observed product attribute
θ_i is consumer specific coefficient
η_{jt} is unobservable (to researchers) aggregate demand shock
ε_{ijt} is iid Type I (maximum) Extreme Value (0,1)

- Assumptions
  - Consumers maximize current period utility
  - $\theta_i \sim MVN\left(\overline{\theta}, \Sigma\right)$
  - $\eta_{jt} \sim N(0, \tau^2)$

Market Share

- Utility: $U_{ijt} = X_{jt}\theta_i + \eta_{jt} + \varepsilon_{ijt}$

- Individual choice probability

$$s_{ijt} = \frac{\exp\left(X_{jt}\theta_i + \eta_{jt}\right)}{1 + \sum_{k=1}^{J} \exp\left(X_{kt}\theta_i + \eta_{kt}\right)}$$

- Market share

$$s_{jt} = \int s_{ijt}\, d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right)$$
$$= h\left(\eta_t \mid \bar{\theta}, \Sigma, X_t\right)$$

- s_t inherits randomness from $\eta_t = \left(\eta_{1t,\dots,}\eta_{Jt}\right)'$

Likelihood

- Can derive density of s_t from density of η_t using Change-of-Variable Theorem

- Density of η_t is

$$\phi\left(h^{-1}\left(s_t \middle| \overline{\theta}, \Sigma, X_t\right) \middle| \tau\right)$$

Normal pdf, $\eta_{jt} \sim N(0, \tau^2)$

- Therefore, density of s_t is

$$\begin{aligned}\pi\left(s_t \middle| \overline{\theta}, \Sigma, \tau, X_t\right) &= \phi\left(h^{-1}\left(s_t \middle| \overline{\theta}, \Sigma, X_t\right) \middle| \tau\right) J_{(\eta_t \rightarrow s_t)} \\ &= \phi\left(h^{-1}\left(s_t \middle| \overline{\theta}, \Sigma, X_t\right) \middle| \tau\right) \left(J_{(s_t \rightarrow \eta_t)}\right)^{-1}\end{aligned}$$

Computing h^{-1}

- h is defined by a system of **J** non-linear equations

$$s_{1t} = \int \frac{\exp\left(X_{1t}\theta_i + \eta_{1t}\right)}{1 + \sum_{k=1}^{J} \exp\left(X_{kt}\theta_i + \eta_{kt}\right)} d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right)$$

$$\vdots$$

$$s_{Jt} = \int \frac{\exp\left(X_{Jt}\theta_i + \eta_{Jt}\right)}{1 + \sum_{k=1}^{J} \exp\left(X_{kt}\theta_i + \eta_{kt}\right)} d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right)$$

- BLP propose an iterative procedure which they prove to be a contraction mapping:

$$\eta_{jt}^{new} = \eta_{jt}^{old} + \ln\left(s_{jt}\right) - \ln\left(h\left(\eta_{1t}^{old}, \ldots, \eta_{Jt}^{old} \mid \bar{\theta}, \Sigma\right)\right)$$

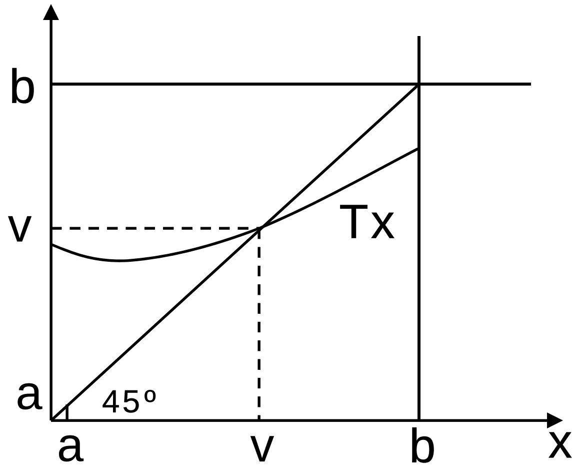
iterate until η_{jt}^{new} and η_{jt}^{old} are close "enough"

Review of contraction mapping

DEFINITION Let $T : S \to S$ be a function mapping S into itself. T is a contraction mapping if $\|Tx - Ty\| < \|x - y\|$, for all $x, y \in S$.

CONTRACTION MAPPING THEOREM

If T is a contraction mapping, then T has exactly one fixed point v in S such that $Tv = v$.



In our model, $T : R^J \to R^J$. BLP prove that $\|T\eta^{new} - T\eta^{old}\| < \|\eta^{new} - \eta^{old}\|$, i.e., we are guaranteed for a fixed point $T\eta = \eta$.

Jacobian does not depend on $\bar{\theta}$ or τ^2

- Jacobian: $J_{(s_t \to \eta_t)} = \left\| \nabla_{\eta_t} s_t \right\|$

  where $$\partial s_{jt} / \partial \eta_{kt} = \begin{cases} \int -s_{ijt} s_{ikt} d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right) & \text{if} \quad k \neq j \\ \int s_{ijt} \left(1 - s_{ijt}\right) d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right) & \text{if} \quad k = j \end{cases}$$

- Re-write utility: $$U_{ijt} = X_{jt} \theta_i + \eta_{jt} + \varepsilon_{ijt}$$ $$= \underbrace{\left(X_{jt} \bar{\theta} + \eta_{jt} \right)}_{\mu_{jt}} + X_{jt} \nu_i + \varepsilon_{ijt}, \quad \nu_i \sim N\left(\mathbf{0}, \Sigma\right)$$

- Elements in Jacobian:

  $$\int f\left(s_{igt}\right) d\Phi\left(\theta_i \middle| \bar{\theta}, \Sigma\right) = \int f\left(\frac{\exp\left(\mu_{jt} + X_{jt} \nu_i\right)}{1 + \sum_{k=1}^{J} \exp\left(\mu_{kt} + X_{kt} \nu_i\right)} \right) d\Phi\left(\nu_i \middle| \mathbf{0}, \Sigma\right) = f\left(\mu_t, \Sigma \mid X_t\right)$$

- But given shares (and covariates), μ_t is determined by Σ
- Thus, given shares (and covariates), Jacobian is only a function of Σ

MCMC: overview

- Recall parameters are: $\Sigma, \bar{\theta}, \tau^2$
- Re-parameterize Σ in terms of r (why?) :

$$\Sigma = U^{'}U, \quad U = \begin{bmatrix} \exp\left(r_{11}\right) & r_{12} & \cdots & r_{1K} \\ 0 & \exp\left(r_{22}\right) & \ddots & \vdots \\ \vdots & \ddots & \ddots & r_{K-1,K} \\ 0 & \cdots & 0 & \exp\left(r_{KK}\right) \end{bmatrix}$$

- Priors

$$\bar{\theta} \sim MVN\left(\bar{\theta}_0, V_{\bar{\theta}}\right)$$

$$r_{jj} \sim N\left(0, \sigma_j^2\right), \ \ r_{jk} \sim N\left(0, \sigma_{off}^2\right), \ j \neq k$$

$$\tau^2 \sim \nu_0 s_0^2 / \chi_{\nu_0}^2$$

- Posterior

$$\propto \underbrace{\prod_t \pi\left(s_t \middle| \bar{\theta}, r, \tau, X_t\right)}_{\text{likelihood}} \times Prior\left(\bar{\theta}, r, \tau\right)$$

More on the re-parameterization

$$\Sigma = U'U = \begin{bmatrix} \exp(2r_{11}) & r_{12}\exp(r_{11}) & r_{13}\exp(r_{11}) & r_{14}\exp(r_{11}) \\ & r_{12}^2 + \exp(2r_{22}) & r_{12}r_{13} + r_{23}\exp(r_{22}) & r_{12}r_{14} + r_{24}\exp(r_{22}) \\ symmetric & & r_{13}^2 + r_{23}^2 + \exp(2r_{33}) & r_{13}r_{14} + r_{23}r_{24} + r_{34}\exp(r_{33}) \\ & & & r_{14}^2 + r_{24}^2 + r_{34}^2 + \exp(2r_{44}) \end{bmatrix}$$

- Priors: $r_{jj} \sim N\left(0, \sigma_j^2\right), \; r_{jk} \sim N\left(0, \sigma_{off}^2\right), \; j \neq k$
- Implied prior variance and mean on diagonals of Σ are (k=1,2,3,4):

$$\text{var}\left[\Sigma_{kk}\right] = 2(k-1)\sigma_{off}^4 + \exp\left(8\sigma_k^2\right) - \exp\left(4\sigma_k^2\right)$$
$$E\left[\Sigma_{kk}\right] = (k-1)\sigma_{off}^2 + \exp\left(2\sigma_k^2\right)$$

- Implied prior variance and mean on off-diagonals of Σ are (k=1,2,3):

$$\text{var}\left[\Sigma_{k,(k+1)}\right] = (k-1)\sigma_{off}^4 + \sigma_{off}^2\exp\left(2\sigma_k^2\right)$$
$$E\left[\Sigma_{k,(k+1)}\right] = 0$$

- Goal: let diagonals of Σ have same prior variance
- So, set $\sigma_1^2 = 0.50666596, \; \sigma_{off}^2 = 1.00001292477193, \; \sigma_2^2 = 0.5019265, \; \sigma_3^2 = 0.4969934, \; \sigma_4^2 = 0.4918498$

$$\Rightarrow \text{var}\left[\Sigma_{11}\right] = \text{var}\left[\Sigma_{22}\right] = \text{var}\left[\Sigma_{33}\right] = \text{var}\left[\Sigma_{44}\right] = 50$$
$$\text{var}\left[\Sigma_{12}\right] = 2.7548, \; \text{var}\left[\Sigma_{23}\right] = 3.7288, \; \text{var}\left[\Sigma_{34}\right] = 4.7021$$
$$E\left[\Sigma_{11}\right] = 2.7548, \; E\left[\Sigma_{22}\right] = 3.7288, \; E\left[\Sigma_{33}\right] = 4.7020, \; E\left[\Sigma_{44}\right] = 5.6744$$
$$E\left[\Sigma_{12}\right] = E\left[\Sigma_{23}\right] = E\left[\Sigma_{34}\right] = 0$$

MCMC algorithm

Two sets of conditional draws

$$r \left| \overline{\theta}, \tau^2, \left\{ s_t, X_t \right\}_{t=1}^{T}, \sigma^2_{r\_diag}, \sigma^2_{r\_off} \right.$$

$$\overline{\theta}, \tau^2 \left| r, \left\{ s_t, X_t \right\}_{t=1}^{T}, \overline{\theta_0}, V_{\overline{\theta}}, \nu_0, s_0^2 \right.$$

1. Random-Walk Metropolis Chain to propose for r

$$r^{new} = r^{old} + MVN\left(\mathbf{0}, \sigma^2 D_r \right)$$

2. Gibbs sampler for $\overline{\theta}$ and τ^2
   - A univariate Bayes regression:

$$\mu_{jt} = X_{jt}\overline{\theta} + \eta_{jt}, \quad \eta_{jt} \sim N\left(0, \tau^2\right)$$

MCMC part of the code…

```
# ------------ (1) Gibbs Sampler for thetabar and taosq --------------------
output=runiregG(y=mu,X=X,XpX=XpX,Xpy=crossprod(X,mu),sigmasq=taosq,
               A=Athetabar,betabar=thetabar0,nu=nu0,ssq=s0sq)
thetabar=output$betadraw
taosq=output$sigmasqdraw

# ------------ (2) Metropolis for r ----------------------------------------
# Random-Walk Chain
rN=r+mvrnorm(1,rep(0,(K*(K+1)/2)),varn_r)

ON=Loglhd(rN,mu,thetabar,taosq)
prior_old=sum(-r[1:K]^2/2/sigmasqR_DIAG)+sum(-r[(K+1):(K*(K+1)/2)]^2/2/sigmasqR_off)
prior_new=sum(-rN[1:K]^2/2/sigmasqR_DIAG)+sum(-rN[(K+1):(K*(K+1)/2)]^2/2/sigmasqR_off)

# Evaluate old r (mu) at new (thetabar,taosq)
eta=mu-X%*%thetabar
llhd_old=sum(log(dnorm(eta,sd=sqrt(taosq))))+OO$sumlogjacob

ratio=exp(ON$llhd+prior_new-llhd_old-prior_old)
alphaS=min(1,ratio) # S stands for Sigma
if (runif(1)<=alphaS) {
   r=rN; OO=ON; ns=ns+1; mu=OO$mu
}
```

Brute-Force log-likelihood code…

```
Loglhd_slow = function(thetabar,r,taosq,mu){
    # Purpose: Evaluate log likelihood. Sigma is re-parameterized as r.

    # (1). Transform r to L, where Sigma=LL'
    L=diag(exp(r[1:K]))
    L[lower.tri(L)]=r[(K+1):(K*(K+1)/2)]

    # (2). At given L, do inversion to get mu. Then compute eta
    temp=invert_slow(L,mu,v,crit,T,H,J,lnactS,indTHJ,indJTH)
    mu  = temp$mu; prob = temp$prob; niter = temp$niter
    eta=mu-X%*%thetabar
```

$$\int s_{hjt}\left(1-s_{hjt}\right)d\Phi\left(\theta_h\right)$$

```
    # (3). Jacobian
    # Form J diagonal elements at each time t
    diagonal=rowMeans(prob*(1-prob)) # TJ by 1 vector
```

$$\int -s_{hjt}s_{hkt}\,d\Phi\left(\theta_h\right)$$

```
    # Form the off diagonal elements
    dd=-prob%*%t(prob)/H # TJ by TJ
    cc=aaa*dd+diag(diagonal)#TJ by TJ matrix: block diagonal
```

$$cct=\nabla_{\eta_t}s_t$$

```
    for (t in 1:T){
          cct=cc[((t-1)*J+1):(t*J),((t-1)*J+1):(t*J)] #(t)th block of cc
          logjacob[t]=-log(abs(det(cct)))
    }
    # (4). Form Log Likelihood
    sumlogjacob=sum(logjacob)
    llhd=sum(log(dnorm(eta,sd=sqrt(taosq))))+sumlogjacob

list(llhd=llhd,mu=mu,niter=niter,sumlogjacob=sumlogjacob)
}
```

Slow inversion code

```
invert_slow =
function(L,mu,v,crit,T,H,J,lnactS,indTHJ,indJTH){

#   Purpose: Invert observed shares S at give L to get mean utility mu's.

    niter=0 # number of iterations taken for the inversion
    munew=mu                         # starting value
    muold=munew/2
    upart=X%*%L%*%v
    while (max(abs((muold-munew)/munew))>crit){

          muold=munew
          num=exp(upart+ muold) # JT by H numerator
          den1=matrix(double(T*H),nrow=T)
          for (t in 1:T){
              den1[t,]=1+colSums(num[((t-1)*J+1):(t*J),]) #T by H
          }
          den=matrix(rep(den1,each=J),ncol=H) #replc each t J times,JT by H
          prob=num/den # JT by H
          sh=t(matrix(rowMeans(prob), nrow=J)) # T by J predicted share
          munew=t(matrix(muold,nrow=J))+log(S)-log(sh) # T by J
          munew=as.vector(t(munew)) # length JT vector
          niter=niter+1
    }
List(mu=munew,prob=prob,niter=niter)
}
```

$$1+\sum_{k=1}^{J}\exp\left(X_{kt}\theta_h+\eta_{kt}\right)$$

profile it (on 1000 iterations)...

```
$by.total
                   total.time total.pct self.time self.pct
rRCLogit_slow          663.68     100.0      0.08      0.0
Loglhd_slow            660.68      99.5      7.96      1.2
invert_slow            513.98      77.4     46.46      7.0
colSums                262.24      39.5    127.60     19.2
exp                     98.34      14.8     98.34     14.8
is.data.frame           81.74      12.3      3.58      0.5
inherits                78.16      11.8     69.46     10.5
matrix                  74.30      11.2     22.68      3.4
as.vector               52.18       7.9     43.52      6.6
diag                    49.26       7.4      8.74      1.3
array                   40.30       6.1     40.26      6.1
%*%                     30.04       4.5     30.04      4.5
+                       29.24       4.4     29.24      4.4
prod                    28.52       4.3     26.04      3.9
log                     25.24       3.8      7.30      1.1
/                       24.14       3.6     24.14      3.6
det                     17.48       2.6      2.02      0.3
*                       14.14       2.1     14.14      2.1
determinant             11.62       1.8      1.98      0.3
t                       11.36       1.7      1.68      0.3
determinant.matrix       9.64       1.5      2.74      0.4
rowMeans                 7.70       1.2      6.42      1.0
:                        7.56       1.1      7.56      1.1
-                        6.50       1.0      6.50      1.0
<                        5.62       0.8      5.62      0.8
>                        4.56       0.7      4.56      0.7
$                        3.34       0.5      3.34      0.5
storage.mode<-           2.96       0.4      0.84      0.1
.Call                    2.30       0.3      2.30      0.3
storage.mode             2.04       0.3      1.14      0.2
length                   1.78       0.3      1.78      0.3
```

Make it faster: Vectorize it!

```
invert =
function(L,mu,v,crit,T,H,J,lnactS,indTHJ,indJTH){

#  Purpose: Invert observed shares S at give L to get mean utility mu's.

   niter=0                         # number of iterations taken
   munew=mu                        # starting value
   muold=munew/2
   upart=X%*%L%*%v
   while (max(abs((muold-munew)/munew))>crit){

         muold=munew
         num=exp(upart+ muold)     # num is JT x H
         dim(num)=NULL             # convert num to JTH vector
         num=num[indTHJ]           # convert num to THJ vector
         dim(num)=c(T*H,J)         # convert num to TH * J matrix
         den=1+rowSums(num)        # TH vector
         prob=num/den              # TH * J matrix
         dim(prob)=NULL            # convert prob to THJ vector
         prob=prob[indJTH]         # convert prob to JTH vector
         dim(prob)=c(J*T,H)        # convert prob to JT * H matrix
         sh=rowMeans(prob)         # JT vector
         munew=muold+lnactS-log(sh)# JT vector
         niter=niter+1
   }
list(mu=munew,prob=prob,niter=niter)
}
```

No loop!

Matrix divided by a vector

More on the re-indexing function

```
JTH_THJ=function(J,H,T){
#
# function to convert and index a vector ordered j by t by h (i.e. j
# varies faster than t than h) into a vector ordered t by h by j
#
ind=double(J*H*T)
cnt=1
for (j in 1:J){
   for (h in 1:H) {
      for (t in 1:T) {
      ind[cnt]=(t-1)*J+(h-1)*(T*J)+j
      cnt=cnt+1
      }
   }
}
return(ind)
}
```

- Similarly, THJ_JTH is a function that converts and indexes a vector ordered t by h by j (i.e. t varies faster than h than j), into a vector ordered j by t by h.

- Pre-compute the two indices:

  ```
  indTHJ=JTH_THJ(J,H,T)
  indJTH=THJ_JTH(J,H,T)
  ```

Eliminate Determinants

- Work on algebra to eliminate “det”:

$$cct = U'U$$

$$|cct| = |U'||U| = \left(\prod_{j=1}^{J} diag(U)\right)^2$$

$$\log\left(|cct|^{-1}\right) = -\log\left(\prod_{j=1}^{J} diag(U)\right)^2$$

$$= -2\sum_{j=1}^{J} \log\left(diag(U)\right)$$

- New code:

```
 for (t in 1:T){
     cct=cc[((t-1)*J+1):(t*J),((t-1)*J+1):(t*J)]   #(t)th block of cc
     logjacob[t]=-2*sum(log(diag(chol(cct))))

     # old code:
     # logjacob[t]=-log(abs(det(cct)))
}
```

profile it again (on 1000 iterations)...

Savings of 57.5%!

```
$by.total
                   total.time total.pct self.time self.pct
rRCLogit               282.08     100.0      0.04      0.0
Loglhd                 279.32      99.0     14.36      5.1
invert                 170.66      60.5     47.84     17.0
exp                     91.66      32.5     91.66     32.5
/                       27.22       9.6     27.22      9.6
diag                    23.56       8.4     10.76      3.8
log                     20.88       7.4      3.96      1.4
crossprod               19.98       7.1     19.30      6.8
sum                     19.38       6.9      0.92      0.3
*                       12.74       4.5     12.74      4.5
chol                    11.12       3.9      2.50      0.9
-                       10.46       3.7     10.46      3.7
rowSums                  8.88       3.1      8.42      3.0
diag<-                   7.12       2.5      7.02      2.5
rowMeans                 6.04       2.1      4.82      1.7
as.matrix                5.26       1.9      4.58      1.6
+                        4.72       1.7      4.72      1.7
.Call                    1.74       0.6      1.74      0.6
nrow                     0.96       0.3      0.86      0.3
is.data.frame            0.96       0.3      0.04      0.0
inherits                 0.94       0.3      0.24      0.1
t                        0.76       0.3      0.14      0.0
as.matrix.default        0.66       0.2      0.58      0.2
t.default                0.62       0.2      0.62      0.2
mvrnorm                  0.62       0.2      0.04      0.0
%*%                      0.58       0.2      0.58      0.2
min                      0.56       0.2      0.52      0.2
:                        0.52       0.2      0.52      0.2
JTH_THJ                  0.52       0.2      0.44      0.2
THJ_JTH                  0.48       0.2      0.42      0.1
runiregG                 0.46       0.2      0.02      0.0
```

GMM

- Berry, Levinsohn & Pakes (1995):

$$\mu_{jt} = X_{jt}\bar{\theta} + \eta_{jt}$$

- Theoretical moments :

$$E\left[Z_t'\eta_t\right] = 0$$

- Sample analog:

$$\hat{m}_T\left(\bar{\theta},\Sigma\right) = \frac{1}{T}\sum_{t=1}^{T} Z_t'\left(\hat{\mu}_t\left(\Sigma\right) - X_t\bar{\theta}\right)$$

- GMM objective:

$$\min_{\bar{\theta},\Sigma}\ \hat{m}_T\left(\bar{\theta},\Sigma\right)' A^{-1}\hat{m}_T\left(\bar{\theta},\Sigma\right)$$

- GMM search can be limited to only Σ, because $\bar{\theta}$ can be concentrated out as follows,

$$\hat{\bar{\theta}}\left(\Sigma\right) = \left(X'ZA^{-1}Z'X\right)^{-1} X'ZA^{-1}Z'\hat{\mu}\left(\Sigma\right)$$

Implementing GMM

- How to form Z?
  - Total # of parameters for GMM: dim(thetabar)+dim(r).
  - Form Z by expanding exogenous variables in X or other instruments into polynomials, exponentials, logarithms, and interact with brand intercepts

- How to form A? Two-step GMM
  - Step 1: Let $A = \frac{1}{T}\sum_{t=1}^{T} Z_t' Z_t$

    Minimize the GMM objective=>obtain the residuals $\hat{\eta}_{jt}^{(1)}$, (1) stands for Step 1.
  - Step 2: Construct a new $A = \frac{1}{T^2}\sum_{t=1}^{T} Z_t' \hat{\eta}_t^{(1)} \hat{\eta}_t^{(1)\prime} Z_t$

    Minimize the GMM objective again.

    After convergence, re-start the optimization routine from the converged estimates to ensure that the optimization converged.

- Take the final converged results=> $\hat{\Sigma}_{GMM}, \hat{\bar{\theta}}_{GMM}, \hat{\eta}_{jt}^{(2)},$ and $\hat{\tau}_{GMM} = sd\left(\hat{\eta}^{(2)}\right)$

A sampling experiment

- J =3 brands, one outside good
- X contains 3 brand intercepts and a uniform[0,1] distributed attribute
- T =300 time periods
- Base parameters

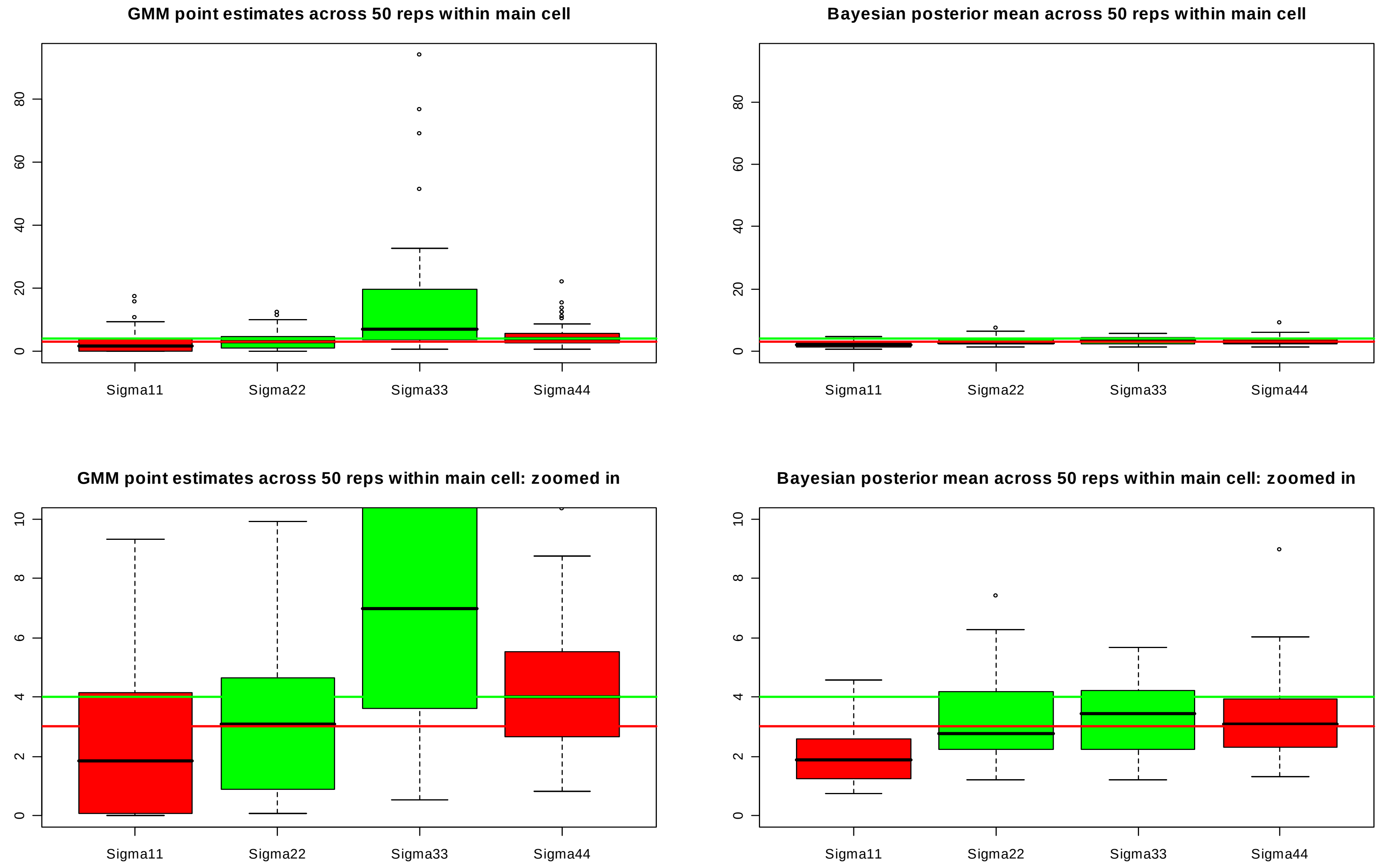
$$\bar{\theta}_{base} = \left(-2, -3, -4, -5\right)$$

$$\Sigma_{base} = \begin{bmatrix} 3 & 2 & 1.5 & 1 \\ & 4 & -1 & 1.5 \\ & & 4 & -0.5 \\ & & & 3 \end{bmatrix}$$

$$\tau^2_{base} = 1$$

- Generate 50 datasets
  - E.g, in one dataset, implied average shares (stdev) over time for each brand: 8.0% (8.5%), 5.1% (4.8%), 1.7% (2.6%)

Results: diagonals of Σ

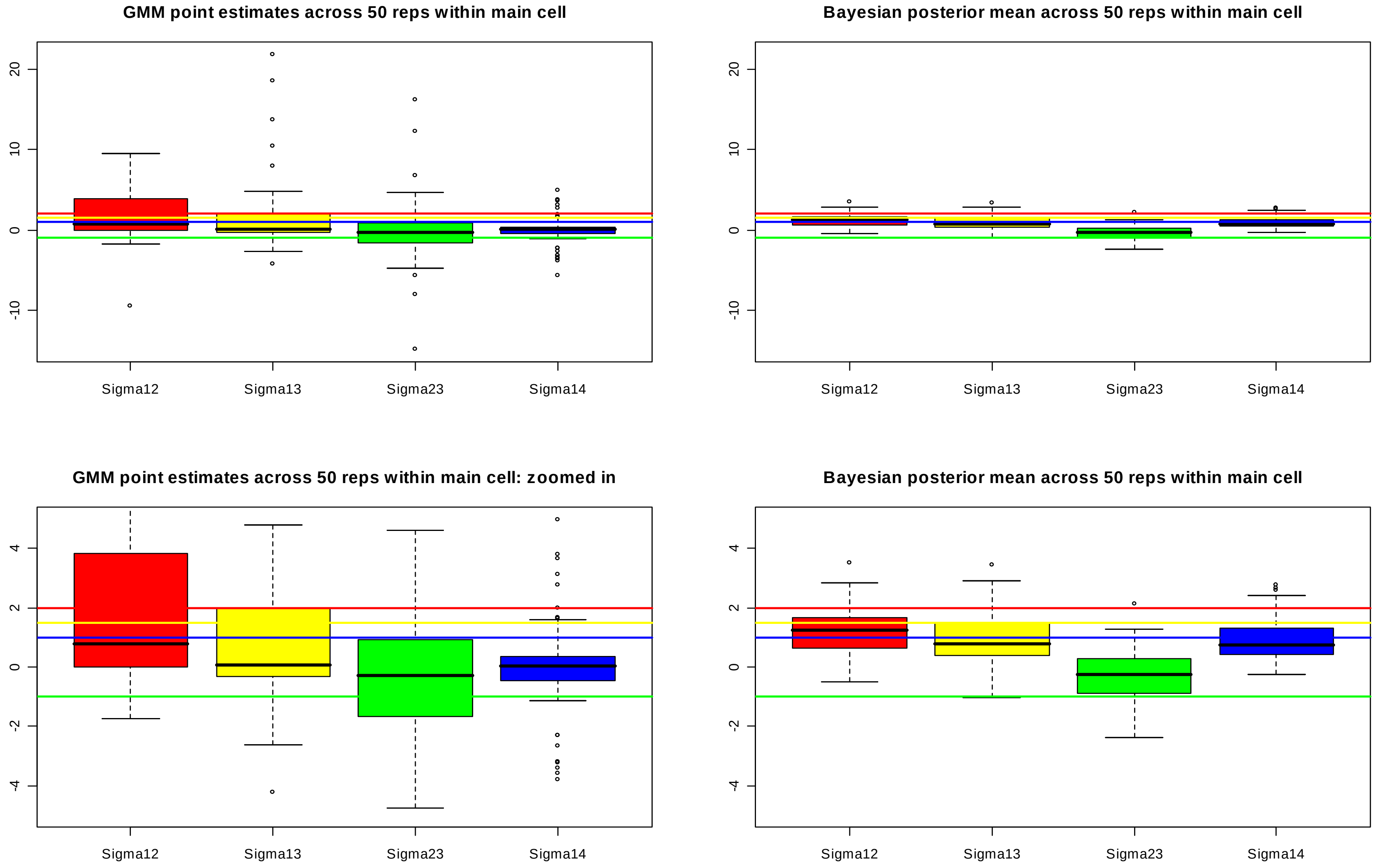

GMM point estimates across 50 reps within main cell

Bayesian posterior mean across 50 reps within main cell

GMM point estimates across 50 reps within main cell: zoomed in

Bayesian posterior mean across 50 reps within main cell: zoomed in

Results: off-diagonals of Σ



GMM point estimates across 50 reps within main cell

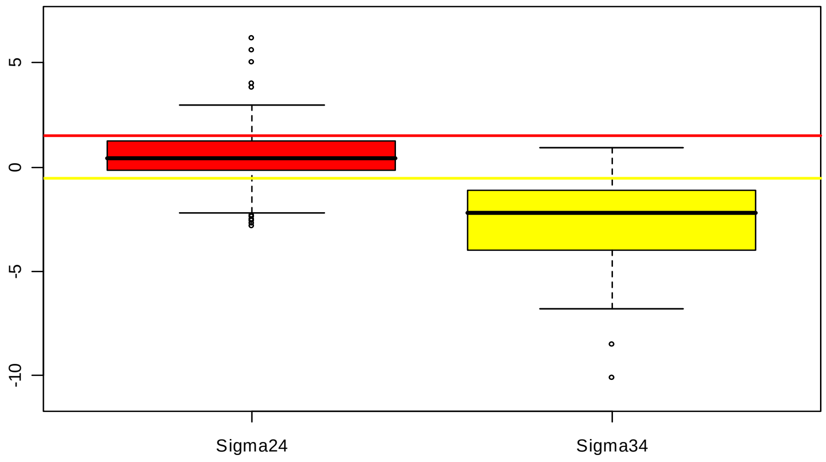
Bayesian posterior mean across 50 reps within main cell

GMM point estimates across 50 reps within main cell: zoomed in

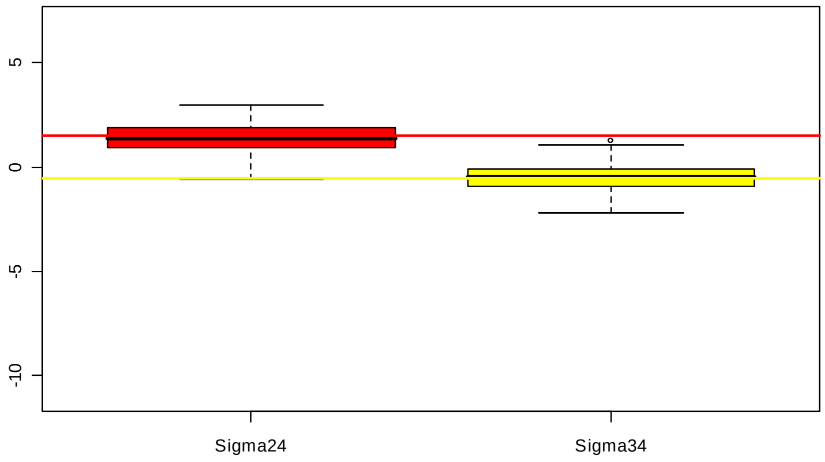
Bayesian posterior mean across 50 reps within main cell

Results: off-diagonals of Σ

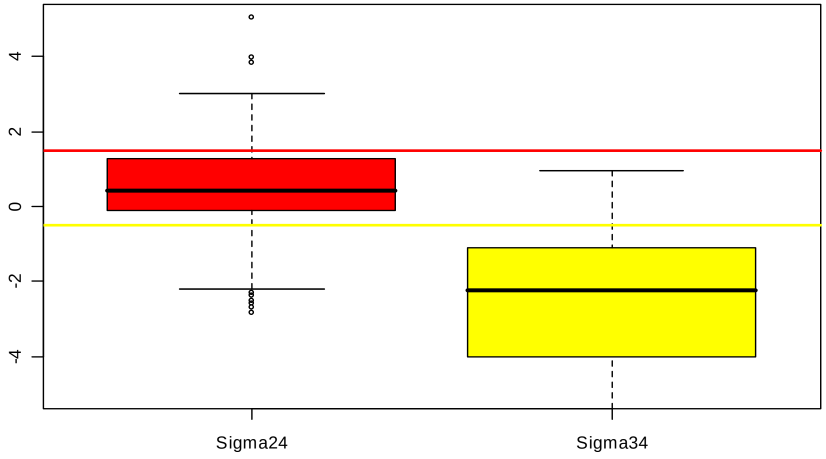
GMM point estimates across 50 reps within main cell



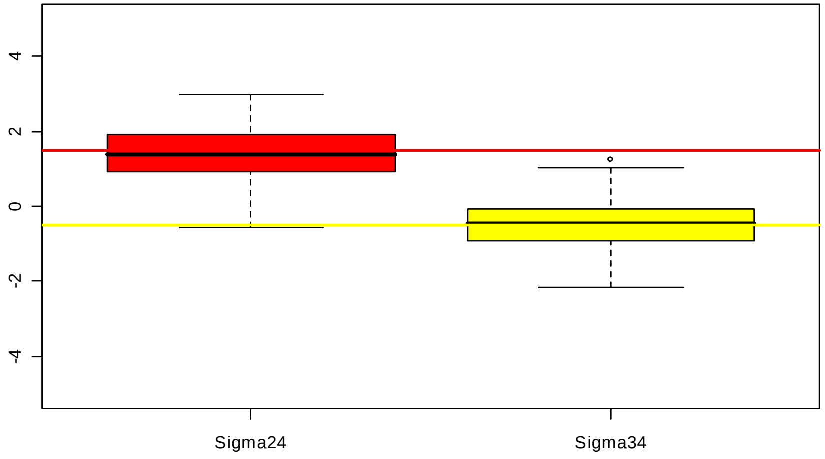
Bayesian posterior mean across 50 reps within main cell



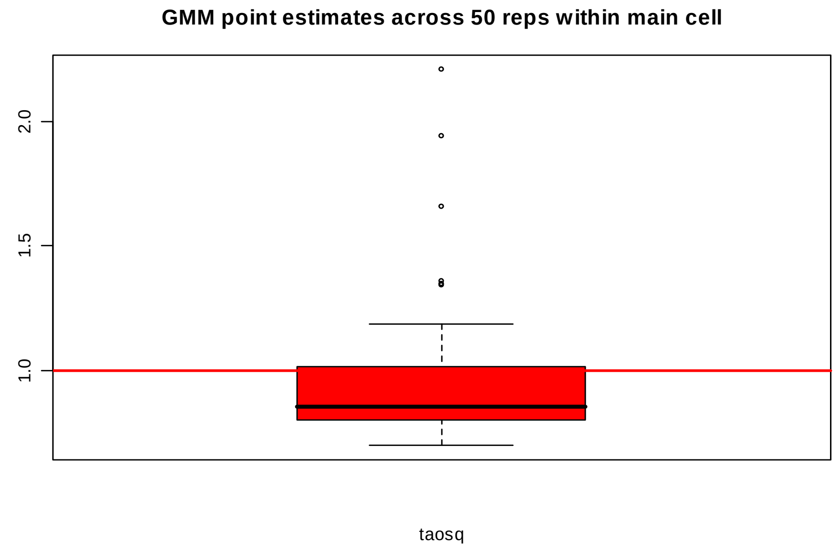
GMM point estimates across 50 reps within main cell: zoomed in

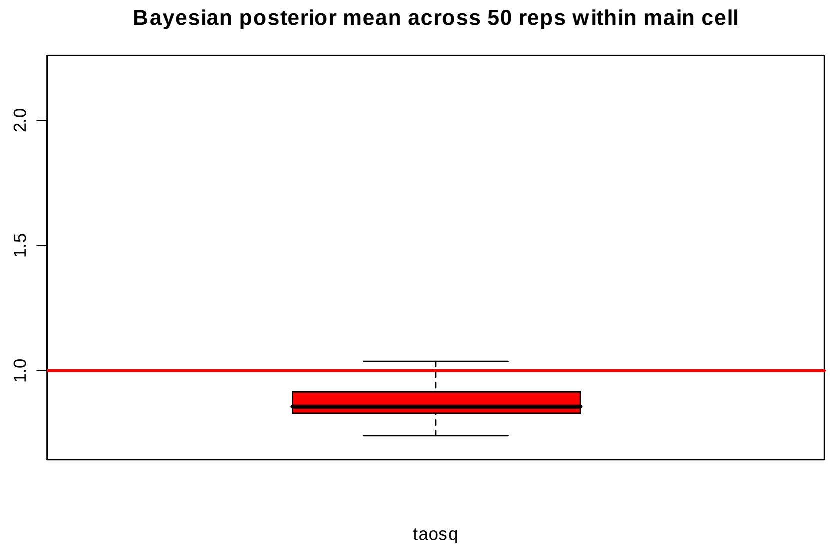


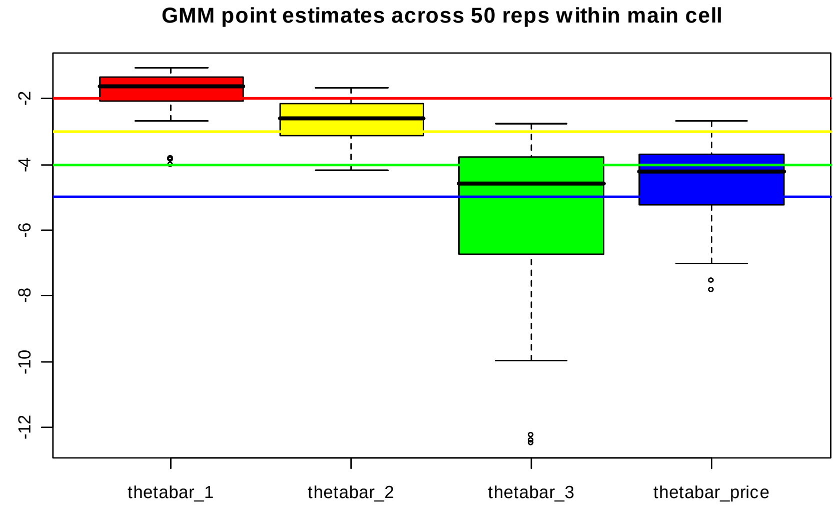
Bayesian posterior mean across 50 reps within main cell

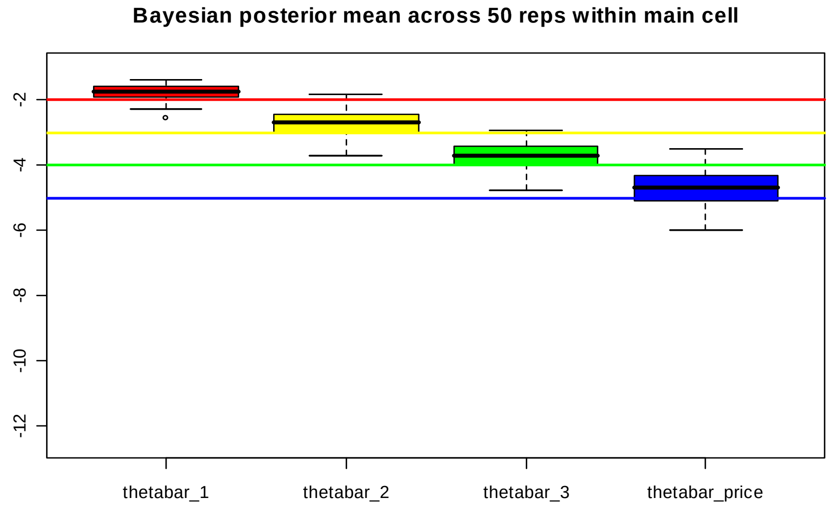


Results: $\tau^2, \bar{\theta}$

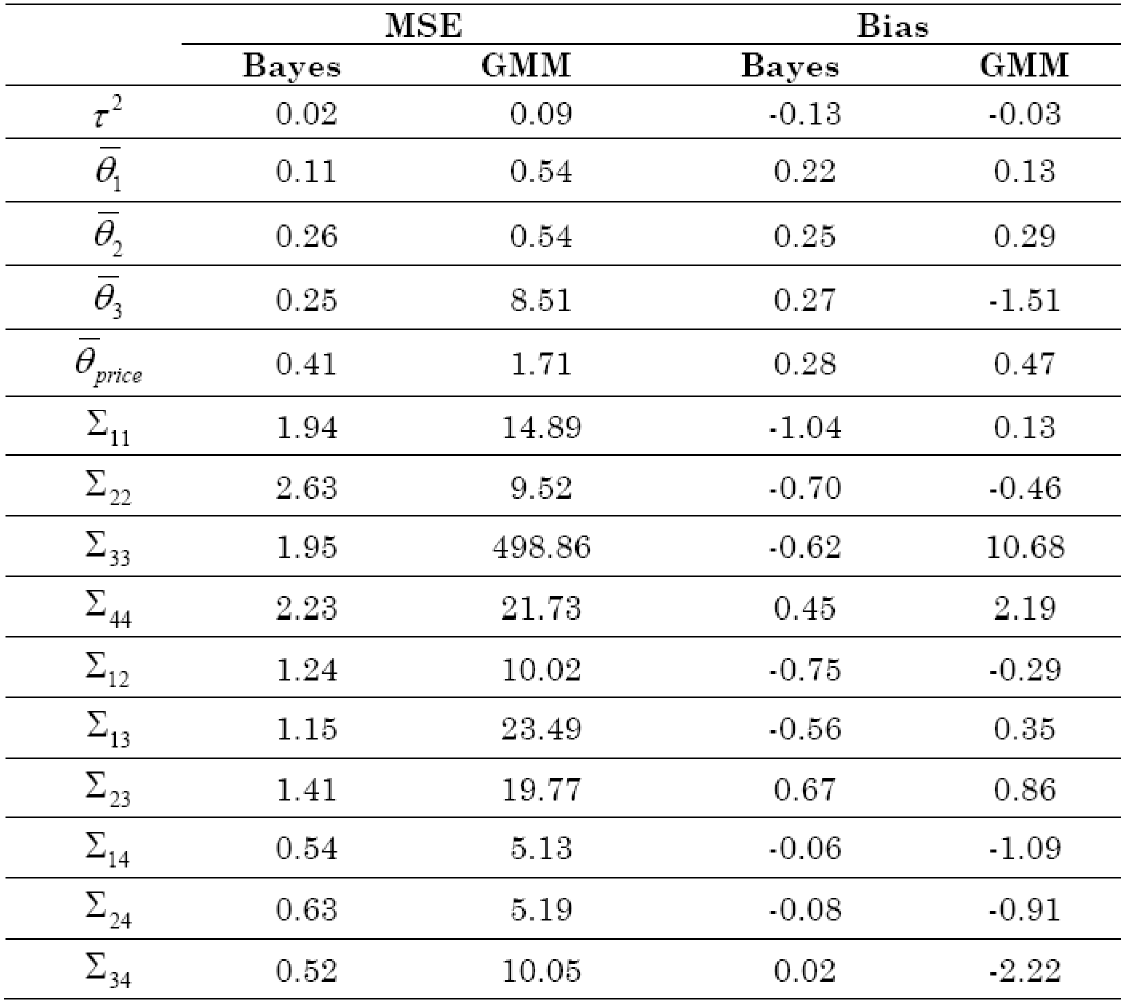

GMM point estimates across 50 reps within main cell


Bayesian posterior mean across 50 reps within main cell


GMM point estimates across 50 reps within main cell


Bayesian posterior mean across 50 reps within main cell

MSE and bias



	MSE		Bias	
	Bayes	GMM	Bayes	GMM
τ^2	0.02	0.09	-0.13	-0.03
$\bar{\theta}_1$	0.11	0.54	0.22	0.13
$\bar{\theta}_2$	0.26	0.54	0.25	0.29
$\bar{\theta}_3$	0.25	8.51	0.27	-1.51
$\bar{\theta}_{price}$	0.41	1.71	0.28	0.47
Σ_{11}	1.94	14.89	-1.04	0.13
Σ_{22}	2.63	9.52	-0.70	-0.46
Σ_{33}	1.95	498.86	-0.62	10.68
Σ_{44}	2.23	21.73	0.45	2.19
Σ_{12}	1.24	10.02	-0.75	-0.29
Σ_{13}	1.15	23.49	-0.56	0.35
Σ_{23}	1.41	19.77	0.67	0.86
Σ_{14}	0.54	5.13	-0.06	-1.09
Σ_{24}	0.63	5.19	-0.08	-0.91
Σ_{34}	0.52	10.05	0.02	-2.22

GMM: 0.7 hr/rep, Bayes: 2.2 hr/rep (Pentium 4, CPU 3 GHz, 1GB RAM)

GMM Estimation: numerical stability

The default setting in GMM estimation

- Optimizer: “optim” in R
- Search algorithm: gradient-based “BFGS” does not work (can’t improve objective value, keeps stopping, etc). Use simplex-based “Nelder-Mead.” All samples “converged” in the sense of meeting conv criterion
- Convergence criterion for simplex search: reltol=1e-6
  - Relative convergence tolerance. Percentage change in objective function < reltol
- Starting value for Sigma: identity matrix
- Convergence criterion for the inversion step: crit=1e-5
- # of heterogeneity draws to evaluate share integral: H=50 (seed=456)

Sensitivity checks on one outlier of GMM estimate

- Start Sigma from true value
- Tighten reltol=1e-10
- Tighten crit=1e-10

Examine one GMM outlier (Rep 29)

- Default

$$\hat{\Sigma} = \begin{bmatrix} 15.1 & 15.5 & 8.2 & 3.0 \\ & 16.3 & 7.6 & 4.7 \\ & & 18.4 & -4.6 \\ & & & 13.1 \end{bmatrix} \quad objval = .298$$

$$\hat{\bar{\theta}} = [-3.5, -4.6, -6.7, -9.6], \quad \widehat{\tau}^2 = 2.2$$

- Start from true

$$\hat{\Sigma} = \begin{bmatrix} 15.7 & 17.2 & 1.7 & -2.6 \\ & 19.6 & 2.0 & -3.2 \\ & & 21.4 & -5.7 \\ & & & 20.5 \end{bmatrix} \quad objval = .179$$

$$\hat{\bar{\theta}} = [-3.6, -5.2, -7.5, -7.9], \quad \widehat{\tau}^2 = 1.7$$

- Tighten reltol to 1e-10:

$$\hat{\Sigma} = \begin{bmatrix} 11.8 & 11.0 & 15.4 & 1.4 \\ & 10.5 & 13.4 & 3.7 \\ & & 28.5 & -6.9 \\ & & & 21.6 \end{bmatrix} \quad objval = .127$$

$$\hat{\bar{\theta}} = [-3.2, -3.8, -8.0, -8.2], \quad \widehat{\tau}^2 = 1.7$$

- Tighten crit=1e-10:

$$\hat{\Sigma} = \begin{bmatrix} 9.6 & 9.9 & 9.7 & 3.7 \\ & 10.6 & 9.1 & 4.6 \\ & & 15.7 & -0.008 \\ & & & 10.8 \end{bmatrix} \quad objval = .348$$

$$\hat{\bar{\theta}} = [-2.8, -3.9, -6.1, -9.1], \quad \widehat{\tau}^2 = 1.6$$

Address mis-specification concerns

- So far, we first generate iid η_{jt} from $N(0,\ 1)$, then conduct inferences assuming $\eta_{jt} \sim N(0,\ \tau^2)$

- Now let's investigate the performance of our estimator in situations where the model is mis-specified.

- Specifically, we generate η_{jt} from as follows, then fit the model assuming $\eta_{jt} \sim N(0,\ \tau^2)$
  - Conditional Heteroskedasticity
  - AR(1): 0.9
  - Asymmetric Beta distribution
  - Symmetric Beta distribution

Mis-specification: Heteroskedasticity

- Instead of homoskedastic shocks, generate shocks from

$$\eta_{jt} \sim N(0,\ V_{jt} \equiv f\left(X_{jt}\right))$$

- To keep things comparable, we require $E\left[V_{jt}\right]=1$
- So,

$$1=E\left[V_{jt}\right]=E\left[\exp\left(a+bP_{jt}\right)\right]=\frac{\exp(a)}{b}\left(\exp(b)-1\right)$$

where $P_{jt} \sim Unif\left[0,1\right]$

- Set b=1, then $a=-\log\left(\exp(1)-1\right)\approx -0.5413$

Mis-specification: AR(1)

- Instead of iid, generate shocks for product j (j=1,2,3) according to

$$\eta_{j,t+1} = \rho\eta_{j,t} + u_{j,t+1} \qquad u_{j,t} \overset{iid}{\sim} N\left(0,\sigma_u^2\right)$$

- We require $\mathrm{var}\left[\eta_{j,t}\right]=1$

- So,

$$1 = \mathrm{var}\left[\eta_{j,t}\right] = \frac{\sigma_u^2}{1-\rho^2}$$

- Set $\rho = 0.9$, we get $\sigma_u \approx 0.4359$

Mis-specification: Asymmetric Beta

- Instead of Normal shocks, generate

$$\eta_{jt} \sim d \cdot Beta\left[\alpha, \beta\right] - c$$

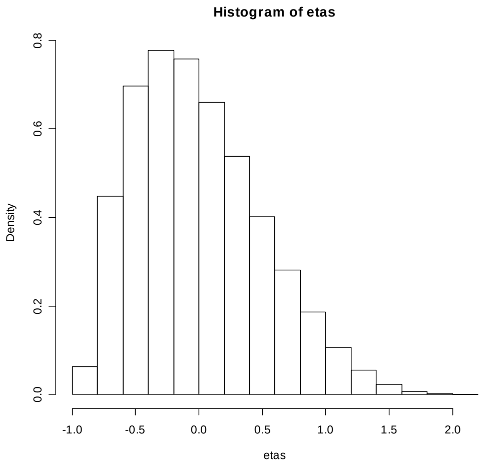
- Set $\alpha = 2, \beta = 5$ to get a logNormal-shaped asymmetric Beta distribution

- We require that η be centered at zero and $sd\left[\eta_{jt}\right] = 0.5$

- So, $0.5 = sd\left[\eta_{jt}\right] = d\sqrt{\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}} = d\sqrt{\frac{2\times 5}{(2+5)^2(2+5+1)}} \quad \Rightarrow d \approx 3.1305$

- Finally, calculate the de-mean factor *c* to keep η centered at zero:

$$c = d \cdot \frac{\alpha}{\alpha+\beta} \approx 0.8944$$



Mis-specification: Symmetric Beta

- Instead of Normal shocks, generate

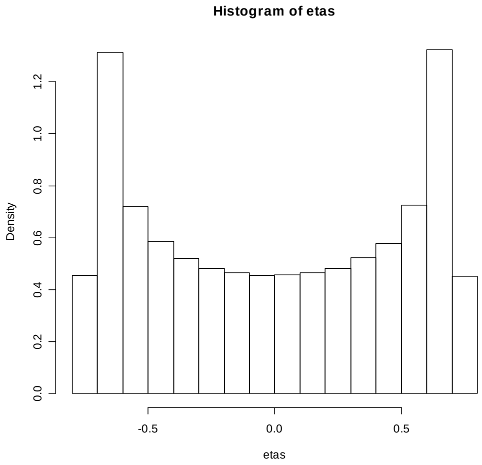
$$\eta_{jt} \sim d \cdot Beta\left[\alpha, \beta\right] - c$$

- Set $\alpha = 0.5, \beta = 0.5$ to get a U-shaped symmetric Beta distribution

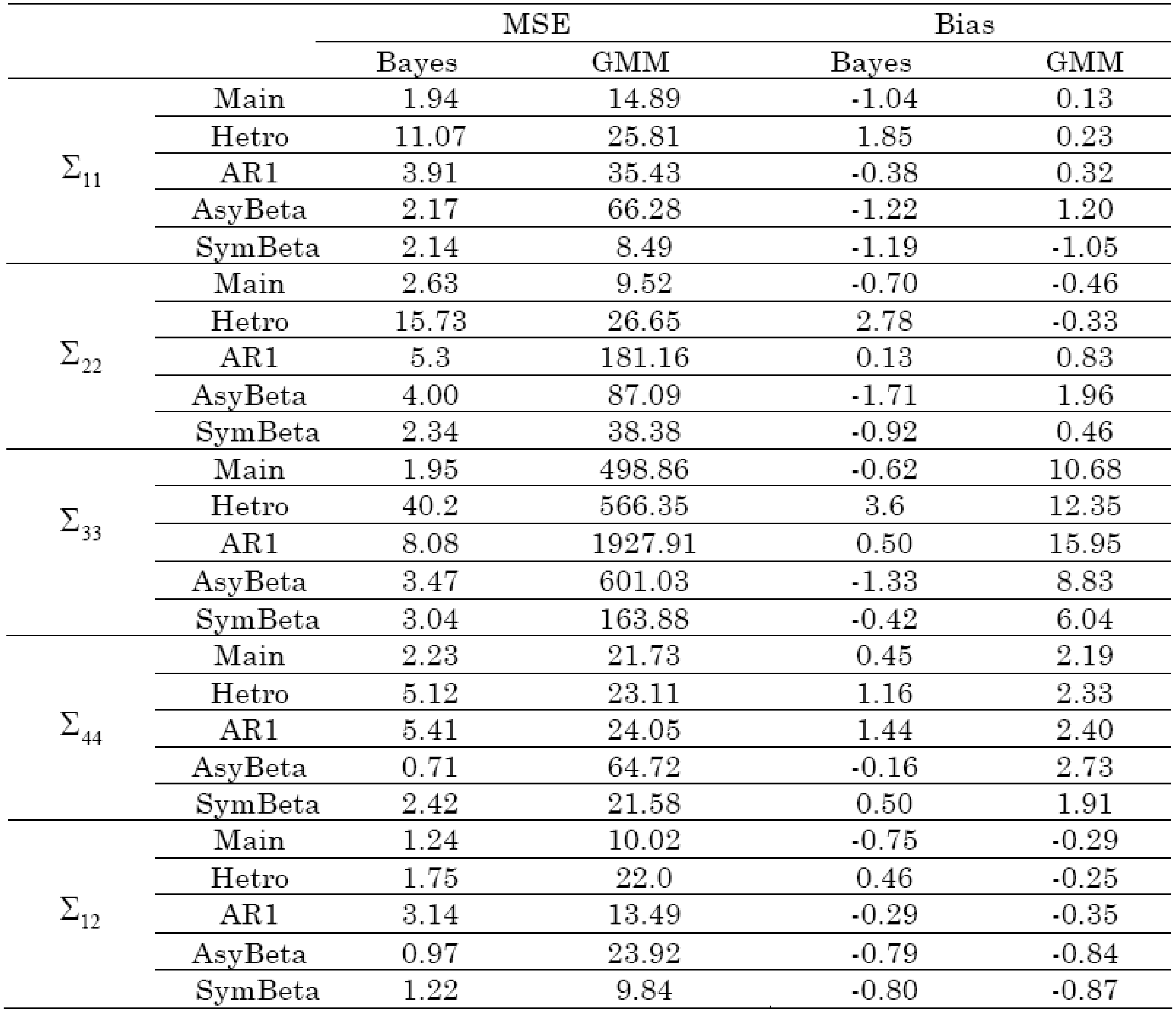
- We require that η centered at zero and $sd\left[\eta_{jt}\right] = 0.5$

- Follow the same procedure as asymmetric Beta, we get

$$d \approx 1.4142,\ c \approx 0.7071$$

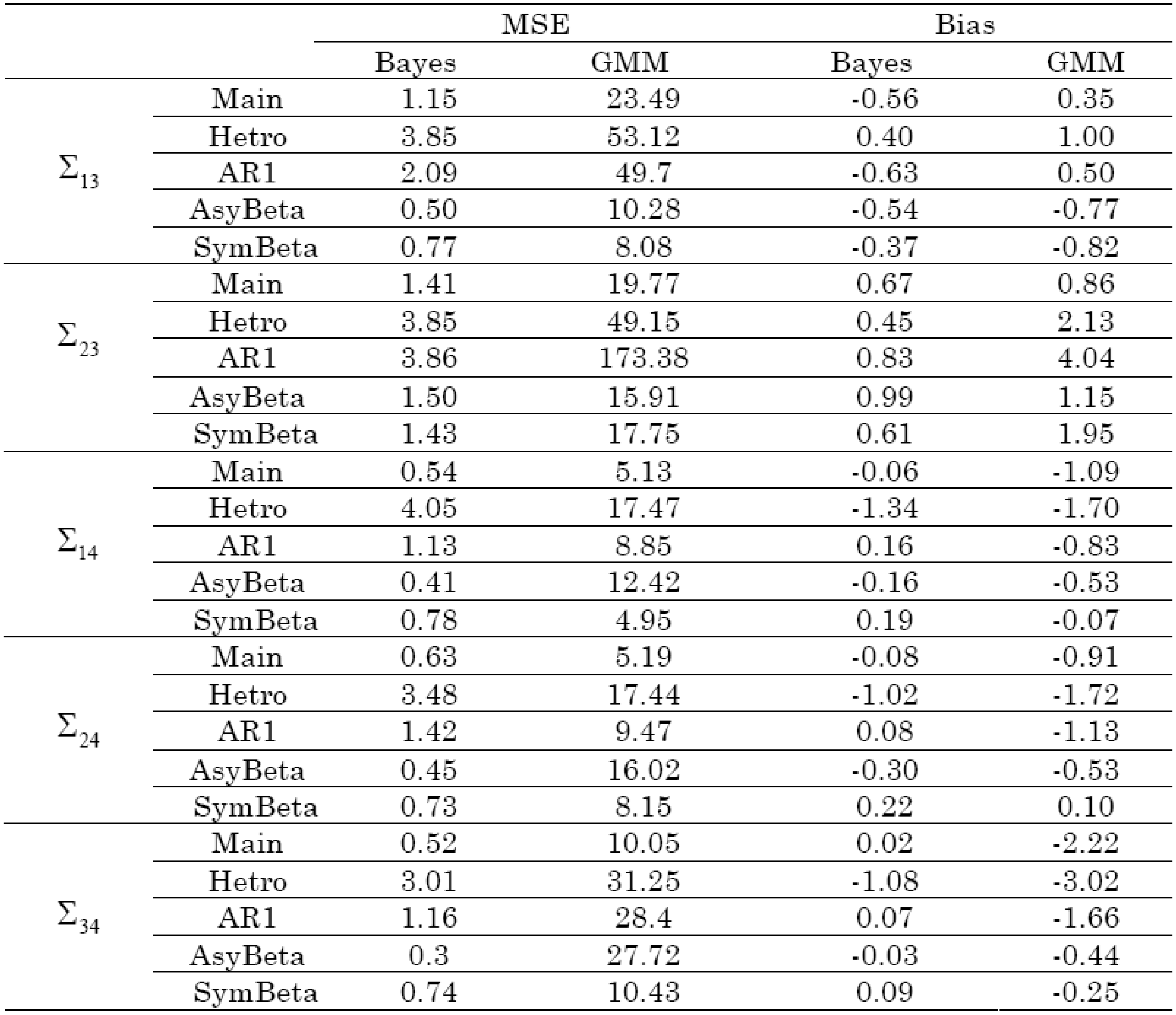


MSE and bias in mis-specified cells



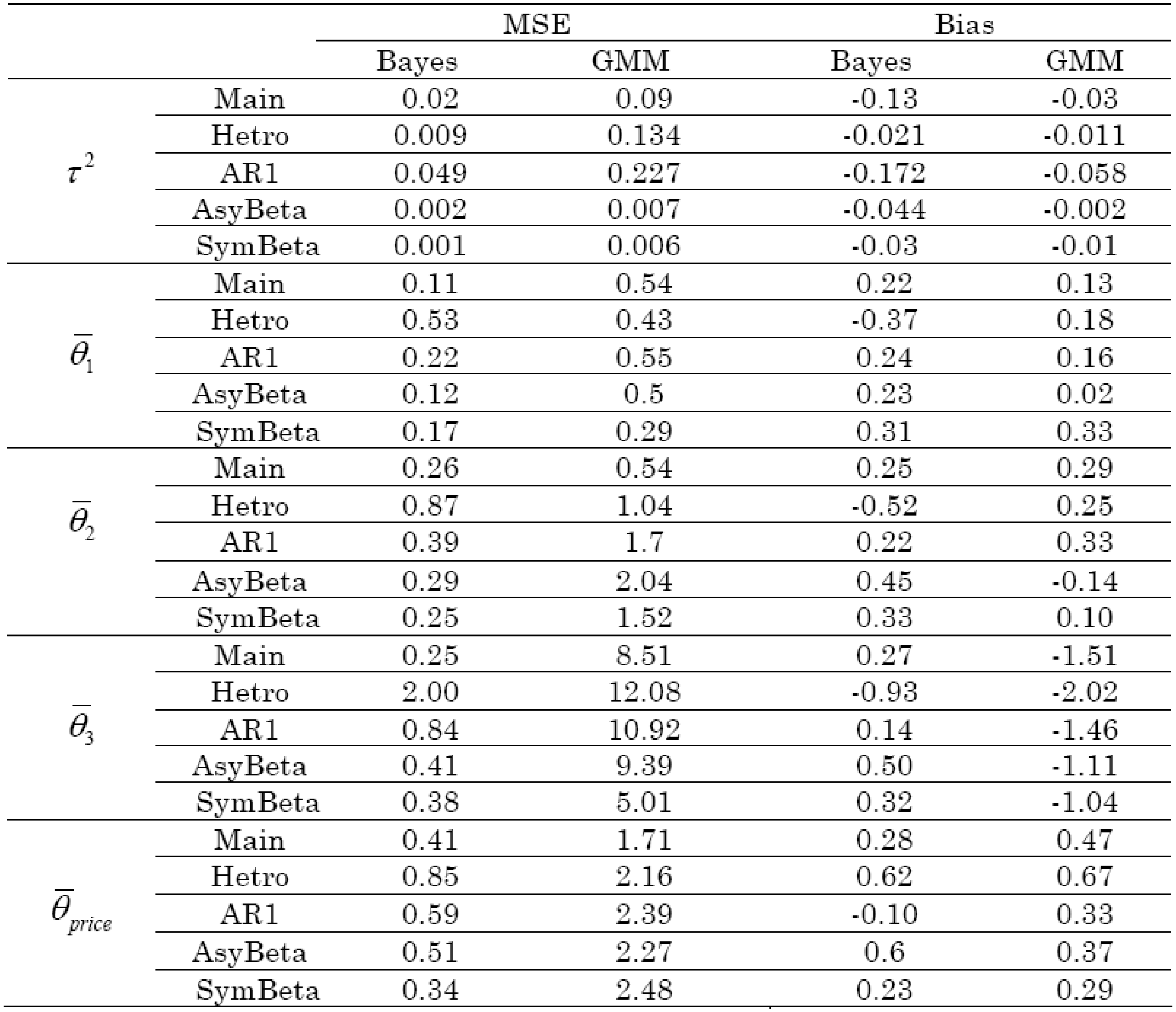
		MSE		Bias	
		Bayes	GMM	Bayes	GMM
Σ_{11}	Main	1.94	14.89	-1.04	0.13
	Hetro	11.07	25.81	1.85	0.23
	AR1	3.91	35.43	-0.38	0.32
	AsyBeta	2.17	66.28	-1.22	1.20
	SymBeta	2.14	8.49	-1.19	-1.05
Σ_{22}	Main	2.63	9.52	-0.70	-0.46
	Hetro	15.73	26.65	2.78	-0.33
	AR1	5.3	181.16	0.13	0.83
	AsyBeta	4.00	87.09	-1.71	1.96
	SymBeta	2.34	38.38	-0.92	0.46
Σ_{33}	Main	1.95	498.86	-0.62	10.68
	Hetro	40.2	566.35	3.6	12.35
	AR1	8.08	1927.91	0.50	15.95
	AsyBeta	3.47	601.03	-1.33	8.83
	SymBeta	3.04	163.88	-0.42	6.04
Σ_{44}	Main	2.23	21.73	0.45	2.19
	Hetro	5.12	23.11	1.16	2.33
	AR1	5.41	24.05	1.44	2.40
	AsyBeta	0.71	64.72	-0.16	2.73
	SymBeta	2.42	21.58	0.50	1.91
Σ_{12}	Main	1.24	10.02	-0.75	-0.29
	Hetro	1.75	22.0	0.46	-0.25
	AR1	3.14	13.49	-0.29	-0.35
	AsyBeta	0.97	23.92	-0.79	-0.84
	SymBeta	1.22	9.84	-0.80	-0.87

MSE and bias in mis-specified cells



		MSE		Bias	
		Bayes	GMM	Bayes	GMM
Σ_{13}	Main	1.15	23.49	-0.56	0.35
	Hetro	3.85	53.12	0.40	1.00
	AR1	2.09	49.7	-0.63	0.50
	AsyBeta	0.50	10.28	-0.54	-0.77
	SymBeta	0.77	8.08	-0.37	-0.82
Σ_{23}	Main	1.41	19.77	0.67	0.86
	Hetro	3.85	49.15	0.45	2.13
	AR1	3.86	173.38	0.83	4.04
	AsyBeta	1.50	15.91	0.99	1.15
	SymBeta	1.43	17.75	0.61	1.95
Σ_{14}	Main	0.54	5.13	-0.06	-1.09
	Hetro	4.05	17.47	-1.34	-1.70
	AR1	1.13	8.85	0.16	-0.83
	AsyBeta	0.41	12.42	-0.16	-0.53
	SymBeta	0.78	4.95	0.19	-0.07
Σ_{24}	Main	0.63	5.19	-0.08	-0.91
	Hetro	3.48	17.44	-1.02	-1.72
	AR1	1.42	9.47	0.08	-1.13
	AsyBeta	0.45	16.02	-0.30	-0.53
	SymBeta	0.73	8.15	0.22	0.10
Σ_{34}	Main	0.52	10.05	0.02	-2.22
	Hetro	3.01	31.25	-1.08	-3.02
	AR1	1.16	28.4	0.07	-1.66
	AsyBeta	0.3	27.72	-0.03	-0.44
	SymBeta	0.74	10.43	0.09	-0.25

MSE and bias in mis-specified cells



		MSE		Bias	
		Bayes	GMM	Bayes	GMM
τ^2	Main	0.02	0.09	-0.13	-0.03
	Hetro	0.009	0.134	-0.021	-0.011
	AR1	0.049	0.227	-0.172	-0.058
	AsyBeta	0.002	0.007	-0.044	-0.002
	SymBeta	0.001	0.006	-0.03	-0.01
$\bar{\theta}_1$	Main	0.11	0.54	0.22	0.13
	Hetro	0.53	0.43	-0.37	0.18
	AR1	0.22	0.55	0.24	0.16
	AsyBeta	0.12	0.5	0.23	0.02
	SymBeta	0.17	0.29	0.31	0.33
$\bar{\theta}_2$	Main	0.26	0.54	0.25	0.29
	Hetro	0.87	1.04	-0.52	0.25
	AR1	0.39	1.7	0.22	0.33
	AsyBeta	0.29	2.04	0.45	-0.14
	SymBeta	0.25	1.52	0.33	0.10
$\bar{\theta}_3$	Main	0.25	8.51	0.27	-1.51
	Hetro	2.00	12.08	-0.93	-2.02
	AR1	0.84	10.92	0.14	-1.46
	AsyBeta	0.41	9.39	0.50	-1.11
	SymBeta	0.38	5.01	0.32	-1.04
$\bar{\theta}_{price}$	Main	0.41	1.71	0.28	0.47
	Hetro	0.85	2.16	0.62	0.67
	AR1	0.59	2.39	-0.10	0.33
	AsyBeta	0.51	2.27	0.6	0.37
	SymBeta	0.34	2.48	0.23	0.29

Inclusion of instrumental variables

- Denote $X_{jt} = \left\{W_{jt}, P_{jt}\right\}$, where price P_{jt} is endogenous:

$$P_{jt} = Z_{jt}\delta + \xi_{jt} \qquad \begin{pmatrix} \xi_{jt} \\ \eta_{jt} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Omega \equiv \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{12} & \Omega_{22} \end{pmatrix}\right)$$

- Joint density of price and share at time *t* is

$$\begin{aligned} \pi\left(P_t, s_t \middle| \overline{\theta}, r, \delta, \Omega\right) &= \pi\left(\xi_t, \eta_t \middle| \overline{\theta}, r, \delta, \Omega\right) J_{(\xi_t, \eta_t \to P_t, s_t)} \\ &= \pi\left(\xi_t, \eta_t \middle| \overline{\theta}, r, \delta, \Omega\right) \left(J_{(P_t, s_t \to \xi_t, \eta_t)}\right)^{-1} \end{aligned}$$

where the Jacobian:

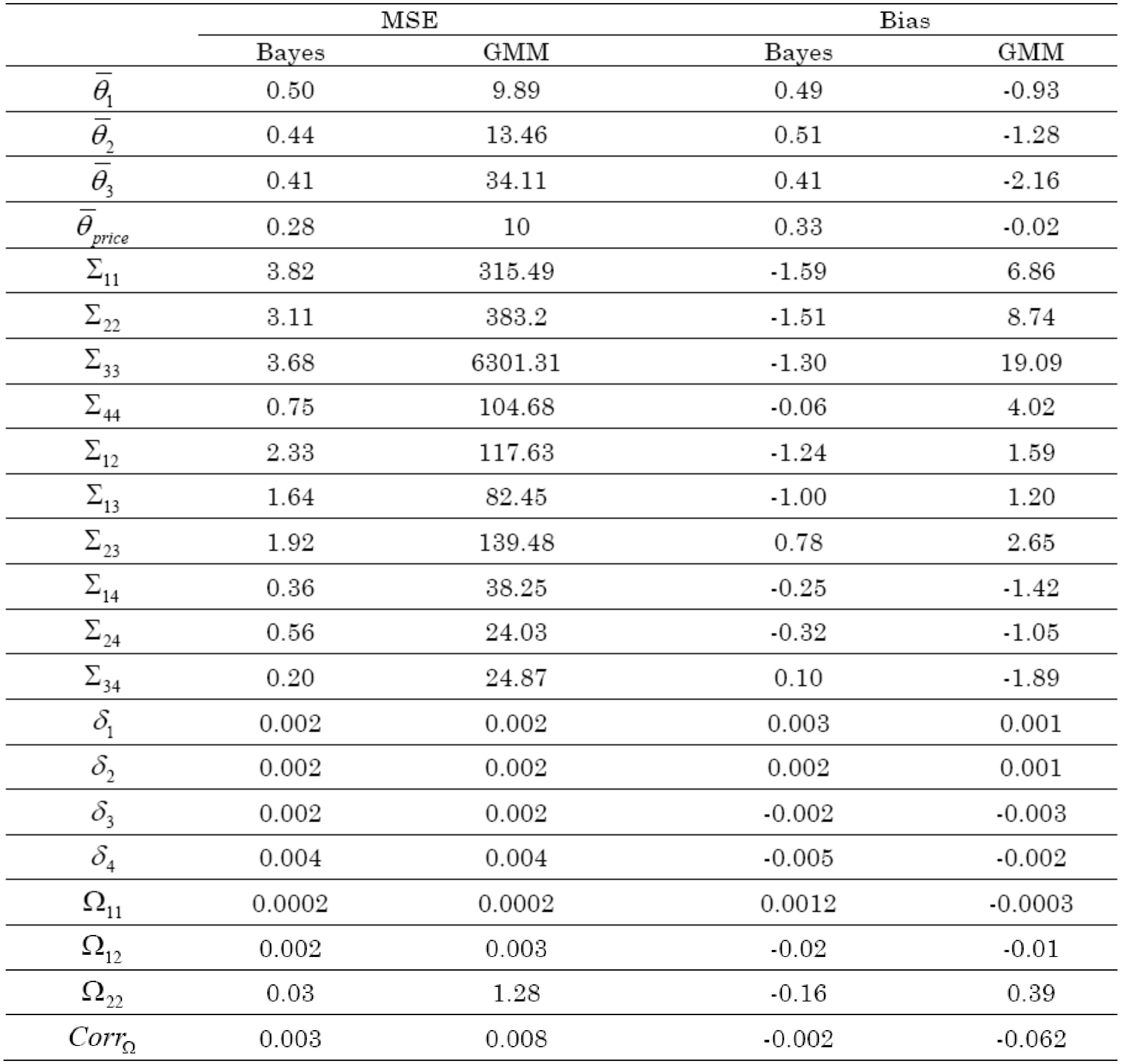
$$J_{(P_t, s_t \to \xi_t, \eta_t)} = \begin{Vmatrix} \nabla_{\xi_t} P_t & \nabla_{\eta_t} P_t \\ \nabla_{\xi_t} s_t & \nabla_{\eta_t} s_t \end{Vmatrix} = \begin{Vmatrix} \boldsymbol{I} & \mathbf{0} \\ \nabla_{\xi_t} s_t & \nabla_{\eta_t} s_t \end{Vmatrix} = \left\|\nabla_{\eta_t} s_t\right\| = J_{(s_t \to \eta_t)}$$

- Two sets of conditionals

$$\overline{\theta}, \delta, \Omega \middle| r, \left\{s_t, X_t, Z_t\right\}_{t=1}^{T}, \overline{\theta}_0, V_{\overline{\theta}}, \overline{\delta}, V_{\delta}, \nu_0, V_{\Omega}$$

$$r \middle| \overline{\theta}, \delta, \Omega, \left\{s_t, X_t, Z_t\right\}_{t=1}^{T}, \sigma_{r\_diag}^2, \sigma_{r\_off}^2$$

MSE and bias in IV cell



	MSE		Bias	
	Bayes	GMM	Bayes	GMM
$\bar{\theta}_1$	0.50	9.89	0.49	-0.93
$\bar{\theta}_2$	0.44	13.46	0.51	-1.28
$\bar{\theta}_3$	0.41	34.11	0.41	-2.16
$\bar{\theta}_{price}$	0.28	10	0.33	-0.02
Σ_{11}	3.82	315.49	-1.59	6.86
Σ_{22}	3.11	383.2	-1.51	8.74
Σ_{33}	3.68	6301.31	-1.30	19.09
Σ_{44}	0.75	104.68	-0.06	4.02
Σ_{12}	2.33	117.63	-1.24	1.59
Σ_{13}	1.64	82.45	-1.00	1.20
Σ_{23}	1.92	139.48	0.78	2.65
Σ_{14}	0.36	38.25	-0.25	-1.42
Σ_{24}	0.56	24.03	-0.32	-1.05
Σ_{34}	0.20	24.87	0.10	-1.89
δ_1	0.002	0.002	0.003	0.001
δ_2	0.002	0.002	0.002	0.001
δ_3	0.002	0.002	-0.002	-0.003
δ_4	0.004	0.004	-0.005	-0.002
Ω_{11}	0.0002	0.0002	0.0012	-0.0003
Ω_{12}	0.002	0.003	-0.02	-0.01
Ω_{22}	0.03	1.28	-0.16	0.39
$Corr_{\Omega}$	0.003	0.008	-0.002	-0.062

Summary of robustness checks

Bayes outperforms GMM in all following sampling experiments (cells):

- Basic cells : $\eta_{jt} \sim N(0, \tau^2)$
  - T=300, τ=1 (main)
  - T=300, τ=2
  - T=300, τ=0.5
  - T=1000, τ=1
- Cells with mis-specified η :
  - Heteroskedasticity: $\eta_{jt} \sim N(0,\ V_{jt} \equiv f\left(X_{jt}\right))$
  - AR(1): 0.9
  - Asymmetric Beta[2,5]: log-normal shaped
  - Symmetric Beta[0.5,0.5]: U-shaped
- Instrumental Variable cell:
  - corr(price shock, demand shock)=0.46,
    instruments explain 32% of total price variation

Conclusions

- The impression that aggregate share models are hard to estimate is partly because of the method used, not the intrinsic property of the model

- Bayesian inference is dramatically more efficient than GMM

- Distribution of aggregate shocks
  - There is concern that Bayes makes more distributional assumptions than GMM. We found that it doesn't matter much
  - Straightforward to extend to a Mixture of Normals. Probably requires more data to identify the shape of this distribution