Nonlinear Pricing without Single Crossing

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OUTLINE

1. **Non-Linear Pricing in Monopoly Market**

2. **An Example With Single Crossing**

3. **An Example Without Single Crossing**

4. **Numerical Explorations**
   - Non-Uniform Distribution of Types
   - Two Dimensional Types
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**General Setup**

- A continuum of consumer with type $\theta \in \Theta$.
- Consumer with type $\theta$ values quantity $q$ by $v(q, \theta)$.
- Monopolist, without being able to observe consumers’ types, charges nonlinear tariff $t(q)$.
- Monopolist’s cost function $C(q)$.

With Revelation Principle, monopolist solve the following problem

\[
\begin{align*}
\text{maximize}_{q,t} & \quad \int_{\theta}^{\bar{\theta}} t(\theta) - C(q(\theta))dF(\theta) \\
\text{subject to} & \quad v(q(\theta), \theta) - t(\theta) \geq 0 \quad \forall \theta \in \Theta \ (\text{IR}) \\
& \quad v(q(\theta), \theta) - t(\theta) \geq v(q(\theta'), \theta) - t(\theta') \quad \forall \theta, \theta' \in \Theta \ (\text{IC})
\end{align*}
\]
Definition: $v_q$ monotonic in types $\theta$. What does it mean?

- Ordering of demands.
- Incentive to lie “downwards”.
- Local incentive constraints imply global incentive constraints (F.O.C. is valid).
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Values: \( v(q, \theta) = \theta \sqrt{q} \), with \( \theta \sim U[2, 3] \) and \( q \geq 0 \).

Cost: \( C(q) = cq \), \( c > 0 \).

Tariff: \( t \geq 0 \).

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\begin{align*}
\text{maximize}_{q,t} & \quad \int_2^3 t(\theta) - C(q(\theta))dF(\theta) \\
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& \quad v(q(\theta), \theta) - t(\theta) \geq v(q(\theta'), \theta) - t(\theta') \quad \forall \theta, \theta' \text{ (IC)}
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AN EXAMPLE WITH SINGLE CROSSING

- **Values:** \( v(q, \theta) = \theta \sqrt{q} \), with \( \theta \sim U[2, 3] \) and \( q \geq 0 \).
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\end{align*}
\]
**Numerical Approach**

We solve this constrained maximization problem numerically.

1. Discretize type space with $N$ grid points, $\theta \in \{\theta_1, \ldots, \theta_N\}$.

2. Reformulate the original problem to the discretized problem

   $\maximize_{q,t} \quad \frac{1}{N} \sum_{i=1}^{N} t(\theta_i) - C(q(\theta_i))$

   subject to
   
   $v(q(\theta_i), \theta_i) - t(\theta_i) \geq 0 \quad \forall i \ (\text{IR})$
   
   $v(q(\theta_i), \theta_i) - t(\theta_i) \geq v(q(\theta_j), \theta_i) - t(\theta_j) \quad \forall i, j \ (\text{IC})$

3. Use KNITRO Active Set Algorithm to solve the discretized problem.

4. Increase $N$ to improve the approximation.
**Figure:** $v(q, \theta)$ under different discretization schemes
Figure: $q(\theta)$ under different discretization schemes
Solutions

**Figure:** Approx. error for $q(\theta)$ under different discretization schemes
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- Values: \( v(q, \theta) = \theta q - \theta^2 q^2 \), with \( \theta \sim U[2, 3] \) and \( q \geq 0 \).
- Cost: \( C(q) = 3q^2 \), \( c > 0 \).
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\end{align*}$$
SOLUTIONS

Discret: 0.5000  Profit: 0.200057

- Quantity
- Tariff
- Profit
- Utility
SOLUTIONS

Discret: 0.1000  Profit: 0.200376

Types $\theta$

<table>
<thead>
<tr>
<th>Discret</th>
<th>Profit</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1000</td>
<td>0.200376</td>
</tr>
</tbody>
</table>

quantity
tariff
profit
utility

Types $\theta$

<table>
<thead>
<tr>
<th>2.0</th>
<th>2.2</th>
<th>2.4</th>
<th>2.6</th>
<th>2.8</th>
<th>3.0</th>
<th>3.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

Non-Linear Pricing  Single Crossing  No Single Crossing  Extensions
Solutions

Discret: 0.0800  Profit: 0.200316

- Type θ
- Quantity
- Tariff
- Profit
- Utility

Types θ
0.00
0.05
0.10
0.15
0.20
0.25
0.30
0.35

Discret: 0.0800  Profit: 0.200316

- Quantity
- Tariff
- Profit
- Utility
SOLUTIONS
Solutions

Discret: 0.0400  Profit: 0.200284
SOLUTIONS

Discret: 0.0200    Profit: 0.195707

- quantity
- tariff
- profit
- utility

Types θ

0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45

Quantity: 20 / 41
Solutions

Discret: 0.0100  Profit: 0.191614

Types θ

0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45

Quantity
tariff
profit
utility

Types θ

0.00 0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45

Solutions
2.6 Conclusion

In our model of nonlinear pricing without the Spence and Mirrlees condition, we showed that the basic assumption of the demand profile approach may not be valid and that the demand profile approach can lead to a suboptimal solution for the monopolist's optimization problem.
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Non-Uniform Distribution of Types

Experiments: Variations of Uniform Discretization

- More mass (grid points) near the beginning.
- More mass near the end.
- More mass at both ends.
- More mass in the middle.
IMPLEMENTATION

- coarse grid distance 0.04, fine grid distance 0.02.
- 64 - 76 variables, 1024 - 1444 constraints
- Multistart option - 100 runs
- KNITRO with Active Set algorithm
- Best solution chosen
General findings

- Similar total profit.
- Similar pooling equilibrium near the end.
- Non-monotonic $q - \theta$ relationship near the beginning.
- Monotonic $u - \theta$ relationship.
NON-UNIFORM DISCRETIZATION OF TYPES

Figure: \( q(\theta) \) for different distributions of types
**Non-Uniform Discretization of Types**

**Figure:** $T(q(\theta))$ for different distributions of types
Non-Uniform Discretization of Types

Figure: $T(q(\theta)) - C(q(\theta))$ for different distributions of types
Non-Uniform Discretization of Types

Figure: $v(q(\theta), \theta) - T(q(\theta))$ for different distributions of types
**TWO DIMENSIONAL TYPES**

- Similar setup
- Values: \( v(q, \theta) = \theta q - \theta^2 q^2 \), with \( \theta \sim U[2, 3] \) and \( q \geq 0 \).
- But now: \( v(q, a, b) = aq - bq^2 \) with \( (a, b) \sim U[2, 3]^2 \)
- Implications:
  - No single crossing again
  - Utility not monotone in types
  - Representation results do not hold - no analytical solution
NON-LINEAR PRICING

SINGLE CROSSING

NO SINGLE CROSSING

EXTENSIONS

UTILITY FUNCTIONS

FIGURE: Utility
**BENCHMARK: FULL INFORMATION**

- Principal maximizes social surplus
- ...and takes it all.

**Figure**: Optimal Supply Schedule
Larger Dimension (d’oh!)
- Number of possible policies explodes
- Cannot rely on local incentive compatibility
- Number of IC constraints explodes

Small feasible region relative to action space
- Hard to find a feasible starting point
- Algorithm often gets stuck
49 grid points
KNITRO with Active Set algorithm
98 variables, 2401 constraints
Multistart option - 50 runs
Best solution chosen
### AMPL Output

**Final objective value:** \( 1.88400489913399 \times 10^1 \)

**Final feasibility error (abs / rel):** \( 5.50 \times 10^{-13} / 7.00 \times 10^{-14} \)

**Final optimality error (abs / rel):** \( 1.01 \times 10^{-08} / 5.78 \times 10^{-09} \)

#### When things go well:

| Iter | Objective | FeasError | OptError | ||Step|| CGits |
|------|-----------|-----------|----------|----------|-------|
| 446  | 1.883564e+01 | 3.600e-08 | 6.101e-01 | 3.056e-02 | 2     |
| 447  | 1.883601e+01 | 3.324e-10 | 6.422e-01 | 1.910e-03 | 6     |
| 448  | 1.883670e+01 | 1.220e-09 | 5.466e+00 | 5.759e-03 | 1     |
| 449  | 1.883723e+01 | 5.542e-09 | 1.527e+00 | 5.040e-03 | 4     |
| 450  | 1.883764e+01 | 9.788e-11 | 6.268e-01 | 5.039e-03 | 1     |
| 451  | 1.883828e+01 | 4.610e-09 | 3.196e-02 | 8.818e-03 | 3     |
| 452  | 1.883923e+01 | 1.554e-08 | 3.980e-02 | 1.764e-02 | 1     |
| 453  | 1.883979e+01 | 1.669e-08 | 4.290e-02 | 1.764e-02 | 1     |
| 454  | 1.883992e+01 | 2.584e-08 | 2.794e-02 | 1.180e-02 | 5     |
| 455  | 1.884004e+01 | 2.525e-08 | 2.867e-02 | 1.179e-02 | 2     |
| 456  | 1.884005e+01 | 8.475e-10 | 1.118e-02 | 4.480e-03 | 9     |
| 457  | 1.884005e+01 | 4.582e-12 | 3.906e-05 | 1.363e-05 | 7     |
| 458  | 1.884005e+01 | 2.220e-15 | 9.310e-08 | 2.196e-08 | 6     |
### When Things Go Bad

| Iter | Objective | FeasError | OptError | ||Step|| | CGits |
|------|-----------|-----------|----------|----------|----------|-------|
| 0    | -4.821871e+01 | 1.046e+01 |          |          |          |       |
| 1    | -6.597676e+00 | 5.046e+00 | 9.596e+03 | 6.716e+00 | 2        |       |
| 2    | 6.938318e+00 | 4.733e+00 | 1.434e+05 | 9.334e+00 | 1        |       |
| 3    | -5.364320e+01 | 3.293e+00 | 7.311e+04 | 1.598e+01 | 2        |       |
| 4    | -7.743043e+01 | 1.711e+00 | 1.033e+05 | 6.115e+00 | 1        |       |
| 5    | -5.054798e+01 | 1.481e-01 | 1.386e+02 | 2.865e+00 | 0        |       |
| 6    | -4.105956e+01 | 7.316e-02 | 4.901e+01 | 8.087e-01 | 0        |       |
| 7    | -3.547193e+01 | 1.743e-02 | 1.398e+01 | 4.493e-01 | 1        |       |
| 8    | -3.303212e+01 | 4.191e-03 | 1.466e+01 | 2.151e-01 | 2        |       |
| 9    | -3.110397e+01 | 5.802e-03 | 2.018e+01 | 2.646e-01 | 1        |       |
| 10   | -2.808023e+01 | 2.913e-04 | 9.807e+00 | 2.665e-01 | 2        |       |
| 11   | -2.663157e+01 | 5.511e-05 | 1.125e+01 | 1.436e-01 | 2        |       |
| 12   | -2.593095e+01 | 1.710e-05 | 4.918e+01 | 7.349e-02 | 2        |       |
| 13   | -2.591876e+01 | 1.687e-05 | 4.895e+00 | 1.148e-03 | 3        |       |
| 14   | -2.591589e+01 | 9.421e-06 | 8.968e+00 | 2.871e-04 | 8        |       |
| 15   | -2.590045e+01 | 3.179e-08 | 9.182e+00 | 1.426e-03 | 1        |       |
| 16   | -2.590009e+01 | 3.119e-08 | 1.756e+00 | 4.975e-05 | 1        |       |
| 17   | -2.589618e+01 | 7.424e-09 | 4.210e+00 | 3.809e-04 | 1        |       |
| 18   | -2.589618e+01 | 2.366e-09 | 8.687e+00 | 1.387e-04 | 3        |       |
| 19   | -2.589313e+01 | 2.970e-09 | 3.071e+00 | 2.774e-04 | 1        |       |
| 20   | -2.588970e+01 | 4.187e-09 | 3.685e+00 | 3.334e-04 | 1        |       |
| 21   | -2.588957e+01 | 4.226e-09 | 1.428e+00 | 1.248e-05 | 1        |       |
| 22   | -2.588956e+01 | 4.412e-09 | 2.508e-02 | 5.990e-07 | 1        |       |
MORE GRAPHS!

**Figure:** Optimal Supply Schedule with Types
**Figure:** Supply w. Types vs Supply with full info
SENSE AND SENSITIVITY

**Figure:** Type (2.6,2.6) - sample IC constraint
Figure: Slack in IR constraint by type (mind the supply distortion!)