THE MODEL

Simple Markov-Perfect Industry Dynamics

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Homogenous Firms

Duopoly

General Analysis

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TWO RELEVANT PRIOR PAPERS

- Abbring and Campbell (Econometrica, forthcoming)
  - Stochastic demand
  - Sunk costs of entry
  - Irreversible exit
  - Homogeneous firms
  - Sequential consinuation decisions
  - Markov-perfect equilibrium
  - Last-In First-Out strategy

- Ericson and Pakes (ReStud, 1995)
  - Investment stochastically improves profitability.
  - Outside good improves exogenously.
  - Outside technology spills over to new entrants.
  - Successful firms eventually coast.
The Model  Homogenous Firms  Duopoly  General Analysis

Required Tools

Static Primitives

Special case: Cournot Competition

- $C$ consumers with demand: $q = 1 - p$.
- Up to two producers with marginal cost $c_H$ or $c_L > c_H$.
- Static profit from each consumer:

$$\pi_H(H) = \left(\frac{1-c_H}{2}\right)^2 > \pi_H(HL) = \left(\frac{1+c_L-2c_H}{3}\right)^2 > \pi_H(HH) = \left(\frac{1-c_H}{3}\right)^2$$

$$\pi_L(L) = \left(\frac{1-c_L}{2}\right)^2 > \pi_L(LL) = \left(\frac{1-c_L}{3}\right)^2 > \pi_L(HL) = \left(\frac{1+c_H-2c_L}{3}\right)^2$$

General case: Producer’s surplus $= C\pi_K(N)$.

- $K \in \{1, \ldots, \bar{K}\}$ and $N = (n_1, n_2, \ldots, n_{\bar{K}})$.
- $\pi_K(N) > \pi_{K-1}(N)$.
- Increasing any element of $N$ strictly decreases $\pi_K(N)$.
- Increasing the $l$'th element of $N$ by 1 and decreasing its $(l-1)$st element strictly decreases $\pi_K(N)$.
**Dynamic Primitives**

Sequential Entry

- Discrete time $t \in \{0, 1, \ldots\}$. Countably many firms.
- Firms have names in $T \times N$.
- In period $t$, $(t, 1)$ is the first firm to make entry decision, $(t, 2)$ is the second, \ldots
- Sunk cost of entry, $\varphi$.
- All entrants have profitability type 1.
- Entry decisions stop when one firm decides to stay out.

Post Entry Evolution

- Irreversible exit. Payoff 0 outside the market.
- $\Pi_{kk'} = \Pr[K_{t+1} = k'|K_t = k]$, $\Pi_{kk'} = 0$ if $k' < k$. (No regress)
- Number of consumers, $C_{t+1} \in [\hat{C}, \check{C}]$, is first-order Markovian.
- Profit $C_{t+1} \pi_{K_{t+1}}(N_{t+1}) - \kappa$. Discount rate $\beta$.

**Timing of Actions**

Start with $(C_{t-1}, N_t)$ → Nature chooses $C_t \sim Q(\cdot|C_{t-1})$ → Active firms receive $C_t \pi_{K_t}N_t - \kappa$ → (t, 1)

$\alpha = 1$; Pay $\varphi$ and enter

$\alpha = 0$; Pass and earn 0

Entry Choices for $(t, 2), (t, 3), (t, 4), \ldots$

All firms make simultaneous continuation decisions.

Nature chooses $K_{t+1}$ using II → Go to next period with $(C_t, N_{t+1})$
PAYOFF RELEVANT HISTORIES

For Potential Entrant \((t, j)\)'s Decision

\[
H_E = \begin{cases} 
C & \text{Demand state} \\
N + j \times (1, 0, 0, \ldots) & \text{Market structure after entry}
\end{cases}
\]

\[H_E \in \mathcal{H}_E \equiv [\hat{C}, \check{C}] \times \mathbb{Z}^K\]

For Continuation Decision After Entry of \(J\) Firms

\[
H_S = \begin{cases} 
C & \text{Demand state} \\
N + J \times (1, 0, 0, \ldots) & \text{Current market structure} \\
K & \text{Type}
\end{cases}
\]

\[H_S \in \mathcal{H}_S \equiv [\hat{C}, \check{C}] \times \mathbb{Z}^K \times K\]

A Markovian strategy is a pair \((A_S(H_S), A_E(H_E))\) for each \(H_S \in \mathcal{H}_S\), and \(H_E \in \mathcal{H}_E\). The strategies are probabilities of survival or entry.

SYMMETRIC MARKOV-PERFECT EQUILIBRIUM

**Definition**

A symmetric Markov-perfect equilibrium is a subgame-perfect equilibrium in which all firms follow the same Markovian strategy.

An incumbent's payoff function is

\[
v_S(H_S) = A_S(H_S)\beta \mathbb{E} \left[ C' \pi_{K'}(N') - \kappa + v_S(H_S'|H_S) \right] | H_S
\]

A potential entrant's payoff function is

\[
v_E(H_E) = A_E(H_E) \left( \beta \mathbb{E} \left[ C' \pi_{K'}(N') - \kappa + v_S(H_S'|H_E) \right] - \varphi \right)
\]

\(\mathbb{E}[:|H_E]\) and \(\mathbb{E}[:|H_S]\) condition on participation next period and on all other firms' strategies.
A DUOPOLY EXAMPLE

- Firms are all identical.
- $\kappa > 0; \varphi > 0; \text{ and } \pi(3) < \kappa.$
- In any symmetric equilibrium, no firm will enter a market with two firms already committed to produce next period.
  
  I. If a third firm does enter, it receives negative profit next period.
  II. Negative profits continue while two rival firms remain.
  III. If any firm exits, by symmetry all firms receive 0.
  IV. Therefore, a third firm’s expected payoff from entry is negative.

DUOPOLY PAYOFF AND ENTRY

- Suppose $N = 2.$
- In any symmetric equilibrium:
  
  I. If one firm receives a positive payoff, both firms do.
  II. Any firm receiving a positive payoff continues.
  III. If a payoff is positive, it must equal
  \[
  \beta \mathbb{E}[C' \pi(2) - \kappa + v(C', 2) | C].
  \]
- Assume (and later verify) that the payoff to continuing alone exceeds the payoff to continuing as a dupolist.
  \[
  v(C, 2) = \max\{0, \beta \mathbb{E}[C' \pi(2) - \kappa + v(C', 2) | C]\}
  \]
- This defines a contraction mapping.
- Strategy of an entrant facing a monopolist
  \[
  A_E(C, 2) = \begin{cases} 
  1 & \text{if } v(C, 2) > \varphi \\
  0 & \text{otherwise.}
  \end{cases}
  \]
**MONOPOLY PAYOFF AND STRATEGY**

- Monopoly equilibrium payoff

  \[ v(C, 1) = \max \{ 0, \beta \mathbb{E}[C' \pi(1) - \kappa + A_E(C', 2) v(C', 1 + A_E(C', 2))] | C \} \]

  This defines a contraction mapping given \( A_E(\cdot, 2) \) and \( v(\cdot, 2) \).

- Strategies of a monopolist and of an entrant facing an empty market

  \[
  A_S(C, 1) = \begin{cases} 
  1 & \text{if } v(C, 1) > 0 \\
  0 & \text{otherwise.} 
  \end{cases}
  \]

  \[
  A_E(C, 1) = \begin{cases} 
  1 & \text{if } v(C, 1) > \varphi \\
  0 & \text{otherwise.} 
  \end{cases}
  \]

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**DUOPOLY SURVIVAL STRATEGY**

\[
A_S(C, 2) = \begin{cases} 
  1 & \text{if } v(C, 2) > 0 \\
  \frac{v(C, 1) - \beta \mathbb{E}[C' \pi(2) - \kappa + v(C', 2)] | C}{0} & \text{if } v(C, 2) = 0, v(C, 1) > 0 \\
  0 & \text{otherwise.} 
  \end{cases}
\]
**DEFAULT TO INACTIVITY**

Potential entrants and monopolists choose inactivity whenever it gives the same payoff as entry or continuation.

**PROPOSITION**

If firms are homogeneous ($\hat{K} = 1$) then there exists a unique symmetric Markov-perfect equilibrium with a strategy that defaults to inactivity.

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**Algorithm for Homogeneous Oligopoly**

A Set $A_E(C, N) = 0$ for $N > \hat{N}$.

B For $N = \hat{N}, \ldots, 1$

B.1 Calculate the fixed point of

$$v(C, N) = \max \{0, \beta E[C' \pi(N) - \kappa + v(C', N + \sum_{j=N+1}^{\infty} A_E(C', j))]\}.$$ 

B.11 Set $A_E(C, N) = I\{v(C, N) > \varphi\}$.

C For $N = \hat{N}, \ldots, 1$

C.1 For $C$ with $V(C, N) = 0$ and $V(C, 1) > 0$, solve for $p(C)$

$$\sum_{j=0}^{N-1} \binom{N-1}{j} p(C)^j (1-p(C))^{N-1-j} E[C' \pi(j+1)-\kappa+v(C', j+1)|C] = 0.$$ 

C.11 Set

$$A_S(C, N) = \begin{cases} 
1 & \text{if } v(C, N) > 0 \\
1/p(C) & \text{if } v(C, N) = 0 \text{ and } v(C, 1) > 0 \\
0 & \text{otherwise.}
\end{cases}$$
Two Types of Firms

- Two profitability types, Low $L = 1$, and High $H = 2$.
- $\Pi_{LH} = \delta$, $\Pi_{LL} = 1 - \delta$, $\Pi_{HH} = 1$, $\Pi_{HL} = 0$
- $N = (n_L, n_H)$
- Assume that a high profitability firm never exits the market when a low profitability rival still continues with positive probability.

Duopoly Equilibrium with Two Types

Start

Value function being calculated

Number of type $H$ firms

Market structure with one firm of each type

Value function in hand

Number of type $L$ firms

Done
**Exit Strategy: High Type Duopolists**

![Graph showing Monopoly payoff $v_S(C, \epsilon_H, H)$ and Duopoly payoff $v_S(C, 2\epsilon_H, H)$]

**Exit Strategy: Low Type Duopolists**

![Graph showing Monopoly payoff $v_S(C, \epsilon_L, L)$ and Duopoly payoff $v_S(C, 2\epsilon_L, L)$]
**Remaining Equilibrium Payoffs**

- Monopoly payoff $v_S(C, \iota_H, H)$
- Duopoly payoff $v_S(C, \iota_H + \iota_L, H)$
- Duopoly payoff $v_S(C, \iota_H + \iota_L, L)$

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**Duopoly Equilibrium Analysis**

**Natural Equilibrium**

No firm ever exits leaving behind a lower type rival. (Cabral, JET 1993, Pakes, Gowrisankaran and McGuire 1993)

**Proposition**

If $\pi(K, 3) < \kappa$ for $K = 1, \ldots, \tilde{K}$, then there exists a unique symmetric and natural Markov-perfect equilibrium with a strategy that defaults to inactivity.
Oriental Lexicographic Algorithm: Setup

A Order market structures lexicographically reading right to left.

\[ N_1 = (0, \ldots, 0, 2) \]
\[ N_2 = (0, \ldots, 1, 1) \]
\[ \vdots \]
\[ N_{(K+2)/2} = (1, 0, \ldots, 0) \equiv \iota \]

B Define \( s = (N, K) \)

C Define \( K(N) = \min\{K|N(K) > 0\} \)

D Define \( S_1 = \{(N_1, \hat{K})\} \)

E For \( j = 2, \ldots, (K+2)/2 \), define

\[ S_j = S_{j-1} \cup \{(N_i, K(N_j))|i \leq j \text{ \& } N_i(K(N_j)) > 0\} \]

Oriental Lexicographic Algorithm: Implementation

A Calculate \( v(C, N_1, \hat{K}) \) as the unique fixed-point to

\[ v(C, N_1, \hat{K}) = \max\{0, \beta \mathbb{E}[C' \pi K(N_1) - \kappa + v(C', N_1, \hat{K})]\}. \]

B For \( j = 2, \ldots, (K+2)/2 \), suppose that \( v(C, s) \) is known for \( s \in S_{j-1} \).

B.1 For any \( s = (N, K) \in S_j/S_{j-1} \) and \( K' < K \) with \( N(K') > 0 \), set

\[ A_S(N, K', C) = I\{v(N, K', C) > 0\}. \]

B.ii Combine the probabilities in II with these strategies to calculate the transition probabilities \( q(s'|s) \) for all \( s' \in S_j \).

B.iii For any \( s = (N, K) \in S_j \), set

\[ A_E(N + \iota, C) = I\{v(N + \iota, 1, C) > \varphi\}. \]

B.iv For all \( s \in S_j/S_{j-1} \), calculate \( v(C, s) \) as the unique fixed point to

\[ v(C, s) = \max\{0, \beta \mathbb{E}[C' \pi K'(N') - \kappa + v(C', N' + A_E(N' + \iota, C), K')]\} \]

C Calculate \( A_S(C, N, K) \) appropriately.
NON-UNIQUE NATURAL SYMMETRIC MPE

At most 3 active firms and 2 profitability types (L and H):

- $\kappa = 4, \varphi = 1, \beta = 0.5, \Pi_{LH} = 0.5$.
- $C_t \in \{C_1 = 0, C_2 = 1e^{-6}, C_3 = 5\}$. Deterministic growth.

<table>
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<th>$\pi_L/\pi_H$</th>
<th>1H</th>
<th>2H</th>
<th>3H</th>
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<td>0L</td>
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<td>/100</td>
<td>/0.9</td>
</tr>
<tr>
<td>1L</td>
<td>99/101</td>
<td>0.89/1.57</td>
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<tr>
<td>2L</td>
<td>1.56/1.58</td>
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Start at $C_1$ with two type $H$ firms in the market. Both firms continuing deters further entry,

$\beta \left( (C_2 \pi_H(2H) - \kappa) + v(C_2, 2H, H) \right) = 246$.

A firm continuing alone will face two entrants,

$\beta \left( (C_2 \pi_H(H) - \kappa) + v(C_2, 1H2L, H) \right) = -1.475$.

Three equilibria, $A_S(C_1, 2H, H) = 1, 0, \text{ and } 5.96e^{-3}$.

GENERAL OLIGOPOLY EQUILIBRIUM ANALYSIS

SEQUENTIALLY PARETO SUPERIOR

No coalition can form a self-enforcing agreement which changes their current choices and thereby strictly increases all their payoffs.

PROPOSITION

There is a modified version of the OLA which always calculates a symmetric and natural Markov-perfect equilibrium that is sequentially Pareto superior.

COROLLARY

If the indifference condition for a mixed-strategy equilibrium always uniquely determines $A_S(C, N, K)$, then the calculated equilibrium is unique.
### Computational Burden

Number of contraction mappings for different $\tilde{N}$, $\tilde{K}$

<table>
<thead>
<tr>
<th>$\tilde{K} / \tilde{N}$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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