Life-Cycle Portfolio Choice, the Wealth Distribution and Asset Prices

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Abstract

In this paper we consider a canonical stochastic overlapping generations economy with sequentially complete markets. We examine how aggregate and individual shocks translate to changes in the distribution of wealth and how these movements in the wealth distribution affect asset prices and the interest rate. We show that effects are generally small if agents have identical beliefs but that differences in opinion lead to large movements in the wealth distribution. The interplay of belief heterogeneity and life-cycle savings motives creates very large movements of asset prices and can potentially generate realistic moments of asset returns.

Keywords: OLG economy, heterogeneous beliefs, life-cycle portfolio choice, wealth distribution, market volatility.

JEL Classification Codes: D53, E21, G11, G12.

1 Introduction

How do aggregate and individual shocks affect the distribution of wealth in an economy and, in turn, how do movements in the wealth distribution affect asset prices and the interest rate? We answer these classical questions in a calibrated model of a stochastic dynamic exchange economy with overlapping generations and complete financial markets. We show that effects are generally tiny if agents have identical beliefs but that differences in opinion lead to considerable movements in the wealth distribution which in turn lead to large asset-price volatility.

We examine a canonical stochastic OLG model with dynamically complete markets and assume that all agents have log-utility. Under this assumption there exists a recursive equilibrium with linear consumption policies and linear pricing functions. This feature enables us to solve models with a large number of generations and substantial intra-generational heterogeneity. In the presence of uncertainty, asset prices depend both on the exogenous shock and the distribution of wealth (at the beginning of the period). If beliefs are identical then the wealth distribution changes little in equilibrium and the resulting impact on asset prices is quantitatively tiny. Differences in beliefs, however, lead to situations where agents place large bets against each other and, as a result, wealth shifts across agents and across generations. Such changes in the wealth distribution strongly affect asset prices.

In our examination of the effects of heterogeneous beliefs we first consider a limiting case with no uncertainty in fundamentals, that is, in endowments and dividends. But agents can buy Arrow securities for different (materially identical) states. If beliefs are identical then the unique long-run equilibrium is a steady state with zero asset-price volatility. For economies with heterogeneous beliefs the predictions of the model are dramatically different. In fact, we prove that for any given stock-price volatility we can construct an OLG economy in which the (unique) equilibrium exhibits at least this volatility. This result holds despite the fact that all agents agree on the distribution of the stock's dividends (it pays one unit of the consumption good in all states).

For an economically intuitive understanding of this result it is important to observe that in OLG models changes in the wealth distribution across generations can lead to very large changes in asset prices. Older generations have a much higher propensity to consume than younger generations and as a result have much stronger incentives to divest of their asset investments. As we would intuitively expect, stock prices will typically be considerably lower when 'old' generations hold most of the stock than when 'young' generations hold most of the stock. Therefore, whether young or old agents hold a majority of the wealth in the economy greatly matters for asset prices. The question then arises under which condition the wealth distribution across generations changes over time. It turns out that differences in beliefs are an important mechanism to generate large changes in the wealth distribution.

This theoretical result raises the question about the magnitude of the asset price volatil-

ity for small differences in beliefs. We answer this question for a realistic calibration of life-cycle income in our model while maintaining the assumption of no uncertainty in dividends and endowments.

We consider several different specifications for beliefs. We start with the case of two exogenous shocks that are i.i.d. and equiprobable. There are two agents per generation who have persistently different beliefs about the shock. Their beliefs are symmetric with respect to the true law of motion. In this model, moderate differences in beliefs lead to large stock-return volatility (about 9 percent quarterly) and a sizable equity premium driven by this volatility (not surprisingly, the market price of risk remains low – markets are complete and there are no constraints on trades). The example is obviously not satisfactory: Agents do not learn about the probabilities and both agents' beliefs remain incorrect even though the true stochastic process is stationary. While we want to remain silent on learning (introducing Bayesian learning in our framework would render the model intractable) we do examine two extensions of the baseline model. First, we consider a specification with three agents, one having the correct beliefs. In this model, if the fraction of agents with the correct belief is relatively small, volatility is still large but the market price of risk turns negative. Volatility decreases as the fraction of agents with correct beliefs becomes larger. However, even when 90 percent of the population has the correct beliefs asset prices can still double due to changes in the wealth distribution caused by the 10 percent of agents with incorrect beliefs. Secondly, we assume that agents typically have identical (and correct) beliefs but that with low probability there can be a regime switch leading to temporary disagreement. There are still 3 agents and a fraction of the population has the correct beliefs while the other two disagree. This specification also leads to high albeit slightly lower volatility. Despite the fact that 2/3 of the time all agents have identical beliefs and the fact that 10 percent of agents always have the correct beliefs, it is easy construct examples where quarterly stock return volatility is above 5 percent.

In order to understand whether there could be other channels that lead to movements in the wealth distribution, we also consider specifications with identical beliefs but with large shocks to endowments and dividends. We show that effects generally are quantitatively small. In fact, we prove that if all endowments are collinear our model exhibits a stochastic steady state and the wealth distribution does not move at all.

There is a large literature on the evolution of the wealth distribution and the effects of the wealth distribution on prices in general equilibrium models. In a model with infinitely lived agents, identical beliefs and complete financial markets there are no movements in the wealth distribution in equilibrium: All shocks are perfectly smoothed out and the wealth distribution as well as prices and choices just depend on the current exogenous shock. If beliefs differ, the wealth distribution changes, but in the long run only the agents with correct beliefs survive (see e.g. Sandroni (2000) and Blume and Easley (2006)). When markets are incomplete these results are no longer true. However, under identical beliefs, in

the stochastic growth model with ex ante identical agents and partially uninsurable income shocks, this does not seem to matter quantitatively. Krusell and Smith (1998) show that in this model macroeconomic aggregates can be almost perfectly described using only the mean of the wealth distribution. Incomplete financial markets alone, therefore, cannot generate movements in asset prices as a result of movements in the wealth distribution (which do not change its mean).

In models with overlapping generations, the distribution of wealth across generations has potentially large effects on stock returns and the interest rate since 'old' agents have a much higher marginal propensity to consume than 'young' agents. This fact was first discovered by Huffman (1987). He points out that a stochastic OLG model can "yield price volatility that would be difficult to rationalize within the context of other models." However, in many specifications of the model, the distribution of wealth moves little in response to aggregate shocks and therefore has a minor effect on aggregate variables. Rios-Rull (1996) shows that the cyclical properties of a calibrated life-cycle model (with identical beliefs) are very similar to the properties of the model with a single infinitely lived agent. Storesletten et al. (2007) consider a model of an exchange economy with incomplete markets and identical beliefs. The fact that their computational strategy yields accurate results shows that in their economy movements in the wealth distribution are also negligible. For the case of complete markets we give a theoretical explanation for these findings as we derive assumptions under which one can prove that the wealth distribution does not vary at all in equilibrium.

The main result of this paper is that relatively small differences in beliefs across generations lead to large movements in the wealth distribution which, in turn, strongly impact aggregate variables. Our model violates the common prior assumption that underlies much of applied general equilibrium modeling. As Morris (1995) points out, this assumption does not follow from rationality. However, any reasonable model that attempts to explain prices in financial markets needs to impose some discipline on the choice of beliefs. Since the focus of this paper is to point out how large the effects of small differences in beliefs could potentially be, we do not present a model which explains these differences. Kurz (1994) develops the theory of rational belief equilibrium to study the effects of heterogeneous beliefs on market volatility. Kurz and Motolese (2001) argue in the context of an OLG economy with two-period-lived agents that belief heterogeneity is "the most important propagation mechanism of economic volatility." Our results support this finding but the underlying economic mechanism in our model with long-lived agents is quite different. In behavioral economics there are various models and explanations for different beliefs, see e.g. Bracha and Brown (2010).

Following Harrison and Kreps (1978), there is a large literature in finance that examines the effects of differences in beliefs and speculation on asset prices and bubbles. This literature has little relation to our paper; in our economy bubbles are impossible (see Santos and Woodford (1997)) and speculation in the sense of Harrison and Kreps (1978) is ruled out by the absence of short-sale constraints. There is also a large literature on the survival and

price impact of noise traders, i.e. agents with wrong beliefs (among many other DeLong et al. (1990), Sandroni (2000), Blume and Easley (2006) and Kogan et al. (2006)). In our economy, if new noise-trader-like agents are born every period, then these have a persistent price impact. The relevant question for us is whether this price impact is quantitatively relevant.

The remainder of this paper is organized as follows. In Section 2 we describe the OLG model and introduce linear recursive equilibria. Section 3 illustrates the main mechanism in the context of a special case that allows for an analytical solution. Section 4 considers model specifications without uncertainty. In Section 5 we discuss the effects of exogenous shocks on asset prices. Section 6 concludes. The appendix contains all proofs and a description of the numerical method.

2 Model

In this section we first describe our model of stochastic overlapping generations economies. Subsequently we show that the unique equilibrium of our OLG model allows for a linear recursive formulation.

2.1 Stochastic OLG economies

Time is indexed by $t = 0, 1, 2, \ldots$. A time-homogeneous Markov chain of exogenous shocks (s_t) takes values in the finite set $\mathcal{S} = \{1, \ldots, S\}$. The $S \times S$ Markov transition matrix is denoted by Π. We represent the evolution of time and shocks in the economy by a countably infinite event tree Σ. The root node of the tree represents the initial shock s_0. Each node of the tree, $\sigma \in \Sigma$, describes a finite history of shocks $\sigma = s^t = (s_0, s_1, \ldots, s_t)$ and is also called date-event. We use the symbols σ and s^t interchangeably. To indicate that $s^{t'}$ is a successor of s^t (or s^t itself) we write $s^{t'} \succeq s^t$. For $\sigma' \succeq \sigma$, we denote the conditional probability of σ' given σ by $\Pi(\sigma'|\sigma)$ (with a slight abuse of notation).

At each date-event H agents commence their economic lives; they live for N periods. An individual is identified by the date-event of his birth, $\sigma = s^t$, and his type, $h = 1, \ldots, H$. The age of an individual is denoted by $a = 1, \ldots, N$; he consumes and has endowments at all nodes $s^{t+a-1} \succeq s^t$, $a = 1, \ldots, N$. An agent's individual endowments are a function of the shock and his age and type alone, i.e. $e^{s^t,h}(s^{t+a-1}) = e^{a,h}(s_{t+a-1})$ for some functions $e^{a,h} : \mathcal{S} \to \mathbb{R}_+$, for all $h = 1, \ldots, H$, $a = 1, \ldots, N$.

Each agent has an intertemporal time-separable expected utility function,

$$U^{s^t,h}(c) = \log\left(c(s^t)\right) + \sum_{a=1}^{N-1} \delta^a \sum_{s^{t+a} \succeq s^t} \pi^{a,h}(s^{t+a}|s^t) \log\left(c(s^{t+a})\right).$$

The discount factor $\delta > 0$ is constant and identical across agents, while the subjective probabilities $\pi^{a,h}(\sigma'|\sigma) > 0$, $\sigma' \succeq \sigma$, may vary with age a and type h. The Markov chain

describing the agents' subjective beliefs[1] may not be time-homogenous and vary with age. In particular it may differ from the "true" law of motion generated by Π.

At each date-event s^t, there are S Arrow securities in zero net supply available for trade. Prices of the Arrow securities are denoted by $q(s^t) \in \mathbb{R}^S$. The portfolio of such securities held by agent (σ, h) is denoted by $\theta^{\sigma,h}(s^t) \in \mathbb{R}^S$. We use subscripts to indicate the Arrow security for a particular shock. The price at node s^t of the Arrow security paying (one unit of the consumption good) at date-event (s^t, s_{t+1}) is denoted by $q_{s_{t+1}}(s^t)$. Similarly, the holding of agent (σ, h) of this security is denoted by $\theta^{\sigma,h}_{s_{t+1}}(s^t)$.

There is also a Lucas tree in unit net supply paying dividends $d(s^t) > 0$. Dividends are a function of the shock alone, so $d(s^t) = d(s_t)$ for some function $d : \mathcal{S} \to \mathbb{R}_{++}$. Let $\phi^{\sigma,h}(s^t)$ denote the holding of individual (σ, h) at date-event s^t and let $p(s^t)$ denote the price of the tree at that node.

Observe that the presence of a complete set of Arrow securities ensures that markets are dynamically complete. It is, therefore, without loss of generality that our economy has only a single Lucas tree since its primary purpose is to ensure that aggregate consumption exceeds aggregate endowments. Thus, we can aggregate the dividends of multiple Lucas trees and restrict attention to a single tree. The aggregate endowment in the economy is $\omega(s^t) = \omega(s_t) = d(s_t) + \sum_{a=1}^{N} \sum_{h=1}^{H} e^{a,h}(s_t)$.

At time $t = 0$, in addition to the H new agents (s_0, h), $h = 1, \ldots, H$, commencing their economic lives, there are individuals of each age $a = 2, \ldots, N$ and each type $h = 1, \ldots, H$ present in the economy. We denote these individuals by (s^{1-a}, h) for $h = 1, \ldots, H$ and $a = 2, \ldots, N$. They have initial tree holdings $\phi^{s^{1-a},h}$ summing up to 1. These holdings determine the 'initial condition' of the economy.

2.2 Sequential competitive equilibrium

The consumption at date-event s^t of the agent of type h born at node s^{t-a+1} is denoted $c^{s^{t-a+1},h}(s^t)$. Whenever possible we write $c^{a,h}(s^t)$ instead. Similarly, we denote this agent's asset holdings by $\phi^{a,h}(s^t)$ and $\theta^{a,h}(s^t)$. This simplification of the notation allows us to use identical notation for the variables of individuals "born" at $t = 0$ and later as well as those of individuals born prior to $t = 0$.

A sequential competitive equilibrium is a collection of prices and choices of individuals

$$\left(q(s^t), p(s^t), \left(\theta^{a,h}(s^t), \phi^{a,h}(s^t), c^{a,h}(s^t) \right)_{a=1,\ldots,N; h=1,\ldots,H} \right)_{s^t \in \Sigma}$$

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[1] We denote the Markov transition matrix for an agent's subjective law of motion by $\pi^{a,h}$. That is, the agent who is currently of age a assigns the probability $\pi^{a,h}(s, s')$ to a transition from the current exogenous state s to the state s' in the next period when he is of age $a+1$. Occasionally it is necessary to refer to multi-step probabilities or to transition probabilities between nodes across the event tree. We denote such probabilities by $\pi^{a,h}(\sigma'|\sigma)$ for nodes $\sigma' \succeq \sigma$. The same convention applies to the "true" law of motion generated by Π.

such that markets clear and agents optimize.

(1) Market clearing equations:

$$\sum_{a=1}^{N-1}\sum_{h=1}^{H}\phi^{a,h}(s^t) = 1, \quad \sum_{a=1}^{N-1}\sum_{h=1}^{H}\theta^{a,h}(s^t) = 0 \quad \text{for all } s^t \in \Sigma.$$

(2) For each s^t, individual (s^t, h), $h = 1, \ldots, H$, maximizes utility:

$$(c^{s^t,h}, \phi^{s^t,h}, \theta^{s^t,h}) \in \arg\max_{c \geq 0, \phi, \theta} U^{s^t,h}(c) \quad \text{s.t.}$$

budget constraint for $a = 1$

$$c(s^t) - e^{1,h}(s_t) + q(s^t) \cdot \theta(s^t) + p(s^t)\phi(s^t) \leq 0,$$

budget constraints for all $s^{t+a-1} \succeq s^t, a = 2, \ldots, N-1$

$$c(s^{t+a-1}) - e^{a,h}(s_{t+a-1}) - \underbrace{\left(\theta_{s_{t+a-1}}(s^{t+a-2}) + \phi(s^{t+a-2})(p(s^{t+a-1}) + d(s_{t+a-1}))\right)}_{\text{beginning-of-period cash-at-hand}} +$$

$$\underbrace{\left(q(s^{t+a-1}) \cdot \theta(s^{t+a-1}) + p(s^{t+a-1})\phi(s^{t+a-1})\right)}_{\text{end-of-period investment}} \leq 0,$$

budget constraint for all $s^{t+a-1} \succeq s^t, a = N$

$$c(s^{t+N-1}) - e^{a,h}(s_{t+N-1}) - \left(\theta_{s_{t+N-1}}(s^{t+N-2}) + \phi(s^{t+N-2})(p(s^{t+N-1}) + d(s_{t+N-1}))\right) \leq 0.$$

The utility maximization problems for the agents (s^{1-a}, h), $a = 2, \ldots, N$, $h = 1, \ldots, H$, who are born before $t = 0$ are analogous to the optimization problems for agents (s^t, h).

The budget equation for agents of age N shows that these agents do not invest anymore but instead consume their entire wealth. As a consequence their portfolios do not appear in the market-clearing equations.

The price of a riskless bond in this setting is simply equal to the sum of the prices of the Arrow securities. We denote the price of the riskless bond by $1/R^f$, where R^f denotes the risk-free rate.

2.3 Linear recursive equilibria

Huffman (1987) considers an OLG economy with incomplete markets, a single Lucas-tree, and logarithmic utility in which agents receive an individual endowment only in the first period of their life. These assumptions lead to a closed-form function for the price of the tree. The tree price depends on the dividends and first-period endowment, both of which are functions of the exogenous shock, and the agents' tree holdings which are the endogenous state variables.

In our OLG model such a closed-form pricing function does not exist. But the assumption of logarithmic utility allows us to express the equilibrium consumption allocations, the price of the Lucas-tree, and the riskless rate as simple functions of state variables. The natural endogenous state variables in the OLG economy are the beginning-of-period cash-at-hand positions of the agents of ages $a = 2, \ldots, N - 1$. Cash-at-hand of agents of age N who are in the last period of their economic lives do not need to be included in the state space. Agents of age $a = 1$ always enter the economy without any initial cash-at-hand. Let $\kappa^{a,h}(s^t)$ denote beginning-of-period cash-at-hand of an individual of age a and type h at node s^t, that is,

$$\kappa^{a,h}(s^t) = \phi^{a-1,h}(s^{t-1})(p(s^t) + d(s^t)) + \theta_{s_t}^{a-1,h}(s^{t-1})$$

for $a = 2, \ldots, N - 1$ and $h = 1, \ldots, H$. The following theorem is proved in the appendix.

THEOREM 1 *Given a shock $s_t = s \in \mathcal{S}$, consumption of the agent of age $a = 1, \ldots, N - 1$, and type $h = 1, \ldots, H$, is a linear function of the individual cash-at-hand positions, that is*

$$c^{a,h}(s^t) = \alpha_{1s}^{a,h} + \sum_{j=2}^{N-1} \sum_{i=1}^{H} \alpha_{jis}^{a,h} \kappa^{j,i}(s^t), \tag{1}$$

for some coefficients $\alpha_{jis}^{a,h} \geq 0$. The price of the tree is also a linear function of the individual cash-at-hand positions, that is

$$p(s^t) = \beta_{1s} + \sum_{a=2}^{N-1} \sum_{h=1}^{H} \beta_{ahs} \kappa^{a,h}(s^t), \tag{2}$$

for some coefficients $\beta_{ahs} \geq 0$. The riskless rate R^f satisfies the relation

$$1/R^f(s^t) = \gamma_{1s} + \sum_{a=2}^{N-1} \sum_{h=1}^{H} \gamma_{ahs} \kappa^{a,h}(s^t), \tag{3}$$

for some coefficients $\gamma_{ahs} \geq 0$.

The three linear functions in the theorem look deceivingly simple. Observe that an agent's cash-at-hand $\kappa^{a,h}(s^t)$ depends on the price of the Lucas-tree $p(s^t)$ whenever he holds a nonzero position of the tree. Equation (2), therefore, is a fixed-point equation instead of a closed-form expression such as the pricing formula in Huffman (1987). Nevertheless the three formulas prove to be very helpful for our analysis because they enable us to compute the OLG equilibrium and to simulate the economy. Unfortunately, we cannot determine the coefficients α, β, and γ analytically unless we make additional assumptions, see Section 3.1 below. We describe how we can compute these quantities numerically in Appendix B.

The state of the economy comprises the exogenous shock $s \in \mathcal{S}$ and the endogenous vector of beginning-of-period cash-at-hand holdings $\kappa \equiv (\kappa^{a,h})_{h=1,\ldots,H; a=2,\ldots,N-1}$. A recursive equilibrium consists of a policy function that maps the state of the economy, (s, κ),

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to current prices and choices as well as a transition function that maps the state in the current period to a probability distribution over states in the subsequent period. We describe the functional form of all of these functions in Appendix B where we introduce the computational method in some detail.

2.4 Aggregate statistics

In our quantitative analysis in Section 4 we examine the asset-pricing implications for realistic calibrations of our model and report estimates for the average values of the first moment and the second central moment of prices and returns. For a given initial state, (s_0, κ_0), the expected value of a stochastic process $(x(s^t))_{s^t \in \Sigma}$ at a fixed date τ is $\sum_{s^\tau} \Pi(s^\tau|s_0)x(s^\tau)$. For a fixed time horizon T we can then define averages over time as follows,

$$E_{s,\kappa}^T(x) = \frac{1}{T}\sum_{t=0}^{T}\sum_{s^t}\Pi(s^t|s_0)x(s^t), \quad Std_{s,\kappa}^T(x) = \sqrt{E_{s,\kappa}^T(x^2) - E_{s,\kappa}^T(x)^2} . \tag{4}$$

To estimate these figures we repeatedly simulate the OLG economy as described in Appendix B. In addition to reporting these moments for a large value for the time horizon T (capturing long-run behavior of the economy) we also report estimates for a fairly small value of T. Specifically, in the calibrated examples in Section 4 where a period is a quarter we report results for time horizons of $T = 100$ and $T = 10000$. One justification for heterogeneous beliefs is the argument that some "structural break" leads to disagreement among agents for some limited time, see e.g. Cogley and Sargent (2008). Obviously, over a short time horizon the average values strongly depend on initial conditions. As we explain below, we will regard the deterministic steady state for an economy with identical beliefs as the most natural initial condition.

3   Some theoretical results

Models with deterministic dividends and endowments serve as a useful benchmark for our analysis. For the discussion in this section and in Section 4 we assume that $e^{a,h}(s) = e^{a,h}$ and $d(s) = d$ for all shocks $s \in \mathcal{S}$. By continuity, the results for such models are similar to those for models with very small shocks to these fundamentals. Thus we view this specification of the general model as a limiting case for economies with little uncertainty.

If in such a model agents have identical beliefs then it is equivalent to a deterministic OLG economy. The economy has a unique steady state, which is independent of beliefs, and for all initial conditions the unique equilibrium converges to this steady state. If agents have differences in beliefs, however, then a steady state does not exist and the wealth distribution changes along the equilibrium path. These changes can have very strong effects on asset prices as we show below.

For comparison, note that in a model with infinitely-lived agents and deterministic endowments and dividends differences in beliefs do not affect asset prices (as long as all

agents have identical time-preferences). Although the wealth distribution may change over time and across shocks all agents agree that the price of the tree should equal the discounted sum of its (safe) dividends.

For our analysis of heterogenous beliefs we need the following Proposition.

PROPOSITION 1 *For given deterministic endowments and dividends, the coefficients α of the consumption functions* (1) *and the coefficients β and γ of the pricing functions* (2) *and* (3) *in Theorem 1 are independent of the specification of beliefs. That is, for given endowments and dividends, the consumption function is*

$$c^{a,h}(s^t) = \alpha_1^{a,h} + \sum_{j=2}^{N-1} \sum_{i=1}^{H} \alpha_{ji}^{a,h} \kappa^{j,i}(s^t), \tag{5}$$

for some coefficients $\alpha_{ji}^{a,h}$, $a = 1, \ldots, N-1$, $h = 1, \ldots, H$, which do not depend on beliefs. The price of the Lucas-tree at any date event s^t is given by an expression of the form

$$p(s^t) = \beta_1 + \sum_{a=2}^{N-1} \beta_a \sum_{h=1}^{H} \kappa^{a,h}(s^t) \tag{6}$$

for some coefficients β_a, $a = 1, \ldots, N-1$, which do not depend on beliefs. Similarly, the risk-free rate R^f satisfies the relation

$$1/R^f(s^t) = \gamma_1 + \sum_{a=2}^{N-1} \gamma_a \sum_{h=1}^{H} \kappa^{a,h}(s^t) \tag{7}$$

for some coefficients γ_a, $a = 1, \ldots, N-1$, which do not depend on beliefs.

Clearly the proposition does not generalize to economies with uncertain dividends. In such economies the beliefs of the agents owning the Lucas-tree matter for its price.

3.1 A benchmark with an analytic solution

We first examine a special case of our OLG model which admits an analytical solution and assume that agents only have positive endowments in the first period of their lives. For notational simplicity, we consider the case $H = 1$ since intragenerational heterogeneity adds little to the results in this section – by Proposition 1, the distribution of wealth within generation plays no role. This allows us to drop the superscript for the type throughout this section. We assume that $e^{a,1} = e^a = 0$, for $a = 2, \ldots, N$ and that $e^1 = 1$. The Lucas-tree pays deterministic dividends $d > 0$. If beliefs are identical, the model is formally identical to a deterministic model without shocks and without financial assets other than the tree. In this model we can solve analytically for the pricing and consumption policies. If beliefs are different across agents it is not clear how to solve the model directly, however Proposition 1 applies and the pricing functions and the consumption functions are the same as in the deterministic model.

The model without uncertainty turns out to be a special instance of the asset-pricing model in Huffman (1987). He also assumes that agents only receive endowments in the first period in their lives and that the only asset available for trade is the tree. While he allows for uncertainty, his result obviously also holds in a deterministic model. Huffman's (1987, p. 142) analysis yields the following coefficients for the linear tree price expression,

$$\beta_1 = \frac{\delta - \delta^N}{1 - \delta^N}, \quad \beta_a = \frac{\delta - \delta^{N-a+1}}{1 - \delta^{N-a+1}}, \quad \text{for } a = 2, \ldots, N-1,$$

for $\delta \neq 1$. Applying L'Hospital's rule as $\delta \to 1$ we obtain for $\delta = 1$ the coefficients

$$\beta_1 = \frac{N-1}{N}, \quad \beta_a = \frac{N-a}{N-a+1}, \quad \text{for } a = 2, \ldots, N-1.$$

For all $\delta > 0$ and all N all coefficients are positive and bounded above by 1.

While Huffman considered an economy with a single tree, in our deterministic economy we can also easily determine the bond-prices. The following corollary of Proposition 1 states the coefficients of the bond price function.

COROLLARY 1 *In the deterministic economy with $e^a = 0$, for $a = 2, \ldots, N$, and $e^1 = 1$, the bond-pricing coefficients γ are*

$$\gamma_1 = \frac{\delta}{(1+d)\sum_{j=0}^{N-1}\delta^j - 1} \quad \text{and} \quad \gamma_a = \frac{\sum_{j=1}^{N}\delta^j}{\left((1+d)\sum_{j=0}^{N-1}\delta^j - 1\right)\sum_{j=0}^{N-a}\delta^j}, \quad a = 2, \ldots, N-1.$$

Given the pricing functions for the bond and the tree, we can now ask how asset prices change with the wealth distribution. For now, we take the wealth distribution as exogenous, in the next subsection we discuss how it changes across time when beliefs are heterogenous.

For the following discussion of the behavior of the prices of the Lucas-tree and the riskless bond we consider the special case $\delta = 1$. This assumption greatly simplifies the formulas. By continuity our qualitative insights carry over to economies with discount factors close to but different from 1.

For $\delta = 1$, Equation (6) implies that the tree price must be

$$p(s^t) = \frac{\frac{N-1}{N} + d\left(\sum_{a=2}^{N-1}\frac{N-a}{N-a+1}\phi^{a-1}(s^t)\right)}{1 - \sum_{a=2}^{N-1}\frac{N-a}{N-a+1}\phi^{a-1}(s^t)}. \tag{8}$$

Suppose the entire tree is held by agents of age $a = 2, \ldots, N-1$. (This cannot happen in equilibrium due to the zero endowment after the first period. However, the argument is also correct but more tedious for a holding of $1 - \epsilon$.) Then the tree price is

$$p(s^t) = (N-a)(1+d) + \frac{a-1}{N}.$$

If the entire tree is held by agents of age N then the price is $p(s^t) = \beta_1 = \frac{N-1}{N}$.

Since $\partial p(s^t)/\partial a < 0$ we observe that the younger the agents holding the entire tree the larger its price. For agents of fixed age a holding the tree and increasing values of N the

tree price grows without bound. If on the contrary the agents of age N hold the entire tree, then its price is equal to $\frac{N-1}{N}$ and thus bounded above by 1. In sum, the price of the Lucas-tree may vary greatly as the wealth distribution changes.

In Appendix A we derive the price of the riskless bond from Equation (7),

$$1/R^f(s^t) = \frac{1}{N(d+1)-1} + \frac{\sum_{a=2}^{N-1}\left(\frac{1}{N-a+1}\right)\phi^{a-1}(s^t)}{1 - \sum_{a=2}^{N-1}\left(1 - \frac{1}{N-a+1}\right)\phi^{a-1}(s^t)}. \tag{9}$$

If the agents of age N have zero holdings of the tree then $\sum_{a=2}^{N-1}\phi^{a-1}(s^t) = 1$ and the price of the riskless bond is constant,

$$1/R^f(s^t) = 1 + \frac{1}{N(d+1)-1}.$$

If the entire tree is held by agents of age N then the price of the riskless bond is $1/R^f(s^t) = \gamma_1 = \frac{1}{N(d+1)-1}$.

Observe that as long as the agents of age N have zero tree holdings the risk-free rate is constant. This fact is perhaps somewhat surprising since the tree price may vary from large values such as $(N-2)(d+1)+\frac{1}{N}$ (if agents of age 2 hold the entire tree) to small values such as $(d+1)+\frac{N-2}{N}$ (if agents of age $N-1$ hold the entire tree). For large ranges of the wealth distribution there in no direct link between the risk-free rate and the price of the Lucas-tree. This is due to the fact that in a deterministic economy agents of age a hold the tree in the current period, agents of age $a+1$ will hold the tree in the next period and the price of the tree will drop. By the absence of arbitrage, the interest rate will be low. This counteracts the effect that the young have a higher propensity to save which causes the tree prices to be high.

Table 1 displays the prices of the Lucas-tree and the riskless bond for an economy in which agents live for $N = 240$ periods. The safe dividend of the tree is $d = 1/2$. The tree

a	2	5	10	100	200	230	239	240
$p(s^t)$	357.00	352.52	345.04	210.41	60.829	15.954	2.4917	0.99583
$1/R^f(s^t)$	1.0028							0.0027855

Table 1: Prices $p(s^t)$ and $1/R^f(s^t)$ if agents of age a hold the entire Lucas-tree

prices varies between 2.4917 and 357.00 without changes in the risk-free rate.

The described price movements in the deterministic economy can only arise if we consider unanticipated shocks to the wealth distribution and even then they are only transitory. The wealth distribution converges quickly to a steady state distribution from any initial condition. Similarly, the tree price and the risk-free rate converge fast to their respective steady-state values. Nevertheless the observed effects prove to be important in our model. In an economy with heterogeneous beliefs the wealth distribution varies endogenously and no steady state exists. As a result large price movements persist indefinitely.

3.2 Differences in beliefs and asset price volatility

If we impose no restrictions on beliefs we can obtain arbitrary movements in the wealth distribution across agents. We can construct beliefs such that in equilibrium, as N or δ become large, the volatility of the tree price (both in the long run and the short run) becomes arbitrarily large while the volatility of the bond price remains arbitrarily low. The following theorem states these facts formally.

THEOREM 2 *Given any tree-price volatility, $\bar{v} < \infty$, and any bond-price volatility, $\underline{v} > 0$, for any time horizon $T > 1$ and any initial condition $\kappa \gg 0$, we can construct an economy where the stock price volatility is at least $\bar{v}$ while the bond-price volatility is at most $\underline{v}$, that is,*

$$Std_{s,\kappa}^T(p) \geq \bar{v}, \quad Std_{s,\kappa}^T(1/R^f) \leq \underline{v} \ .$$

The proof in the appendix constructs economies with $\delta = 1$, letting N become arbitrarily large. In light of the benchmark case above, we can either hold N fixed and choose δ and $(\pi^a)_{a=1,\ldots,N-1}$ or we can hold $\delta \geq 1$ fixed and choose N and $(\pi^a)_{a=1,\ldots,N-1}$ in order to obtain the desired stock-price volatility. The key idea of the proof is to construct beliefs such that most of the wealth is alternately held either by agents of age 2 or by agents of age $N - 2$. The analysis of the benchmark case revealed that the stock price is very large when the young agents hold most of the wealth while it is very small when the old agents hold most of the wealth. In addition, beliefs must be constructed so that agents of age $N - 1$ never buy much of the tree and thus the price of the riskless bond remains almost constant.

The result accentuates that differences in beliefs can have potentially huge effects on the price of the long-lived asset in this economy. If we can freely choose beliefs over the exogenous shocks then we can generate arbitrary price volatility. The price of the tree can move arbitrarily far away from the discounted present value of its dividends if these are discounted using the current interest rate. Following Harrison and Kreps (1978) there is now a large literature in finance that demonstrates how asset pricing bubbles can arise from differences in beliefs and speculation. It is important to note that in our model there can never be bubbles in equilibrium, see Santos and Woodford (1999). Nevertheless, the economy exhibits large swings in the price of the tree which could not be distinguished from an asset pricing bubble if we only examined prices and observed aggregate variables. In an OLG model, movements in the wealth distribution can lead to large changes in the prices of long-lived assets without changing the short-term interest rate.

These theoretical results raise the question on the quantitative importance of small differences in beliefs in an otherwise realistically calibrated economy. Before we answer this question in Section 4, it is interesting to note that equilibrium price volatility in this economy relies crucially on the existence of a rich asset structure.

3.3 Incomplete vs. complete markets

In an OLG economy with a single tree but no other securities the pricing formula for the tree remains the same as in our OLG model. As the analysis in Huffman (1987) shows, there is a steady state with no trade even if beliefs are heterogeneous. In Huffman's economy, agents' consumption and savings decisions are independent of their beliefs, they depend only on the discount factor δ and the age of an agent. An agent of age a always consumes a fixed fraction of his cash-at-hand, no matter what his expectations are for future prices. Therefore, in the absence of Arrow securities there is no complex trading in this economy and zero price volatility in equilibrium in the long run – for any beliefs and discount factors. On the contrary, when there is a complete set of Arrow Securities available for trade as in our OLG model, price volatility can be arbitrary. In this sense, a rich set of financial assets can lead to a huge increase in the volatility of the price of the tree.

4 Changes in the wealth distribution in calibrated examples

The income profile in the OLG model of the previous section is obviously unrealistic. We are, therefore, confronted with the question whether the large asset price volatility in this model is purely a theoretical artifact or instead of economic importance. In this section we consider a version of the model with a realistically calibrated labor income process. It is empirically well documented that observed income processes are "hump-shaped" and clearly we expect this shape to affect agents' investment decisions. Knowing that their income will rise in the future, young agents now have a higher propensity to consume than in the previous model. We would, therefore, expect that fluctuations in the wealth distribution lead to quantitatively smaller fluctuations in asset prices. But how much smaller? Or, put differently, are small differences in beliefs (still) an important source of asset price volatility in a realistically calibrated model?

For the investigation of this issue we consider a specification of the model with calibrated labor income but maintain the assumption that endowments and dividends are deterministic. A time period is meant to represent a quarter and so we assume that agents live for $N = 240$ periods. We use the parameter values estimated by Davis et al. (2006) for a realistic calibration of life-cycle income. They follow the estimation strategy of Gourinchas and Parker (2002) and fit a 5th order polynomial to match average income from the Consumer Expenditure Survey (CEX) and the Panel Study of Income Dynamics (PSID). The resulting age-income profile is given by

$$
\begin{aligned}
log(e^a) \quad = \quad & 6.62362 + 0.334901(\frac{a}{4} + 20) - 0.0148947(\frac{a}{4} + 20)^2 + 3.63424 \cdot 10^{-4}(\frac{a}{4} + 20)^3 \\
& - 4.41169 \cdot 10^{-6}(\frac{a}{4} + 20)^4 + 2.05692 \cdot 10^{-8}(\frac{a}{4} + 20)^5
\end{aligned}
$$

for $a \leq 4 \cdot 43 = 172$ and $e^a = \frac{e^{172}}{2}$ for $a = 173, \ldots, 240$. This profile is hump-shaped with a replacement rate at retirement of 50 percent. We normalize aggregate endowments to be

$\omega = 2$ and assume that the stock's dividends are always $d = 1$.

We examine models with three different belief specifications. In the first specification of the model there are two different types of agents. All agents have constant beliefs across time and type 1 agents and type 2 agents disagree at all times. In the second version of the model there are three agents. Agents 1 and 2 have beliefs as in the first specification, but now there is also a third agent who has correct beliefs at all times. Finally, we consider the case where in "normal" times all three agents agree on the probabilities. However, with low probability regime switches can occur which then lead to temporary disagreement among the agents of types 1 and 2.

4.1 Persistent subjective beliefs

Throughout this first specification of the model, we assume that there are 2 shocks, $s = 1, 2$, which are i.i.d. and equiprobable, i.e. the data-generating Markov chain is given by $\Pi(1,1) = \Pi(1,2) = \Pi(2,1) = \Pi(2,2) = 1/2$.

Using micro-data, Gourinchas and Parker (2002) estimate the annual discount rate to be around 0.97. This figure corresponds to a quarterly discount factor of 0.9924. Alternatively, we can choose δ to match the average riskless rate (of about 1 percent p.a.). We report the risk-free rate from our specifications below and see that for many specifications we need a value of δ above 1 to match the interest rate. Thus we vary agents' discount factor and examine values of δ in $\{0.99, 0.996, 1.0, 1.005\}$. For the specification of beliefs, we assume that both agents believe (correctly) that the process is i.i.d. But all agents of type 1's beliefs satisfy

$$\pi^{a,1}(1,1) = \pi^{a,1}(2,1) = 1/2 + \epsilon, \ \pi^{a,1}(1,2) = \pi^{a,1}(2,2) = 1/2 - \epsilon, \quad a = 1, \ldots, N - 1$$

while agents of type 2 have the following probabilities,

$$\pi^{a,2}(1,1) = \pi^{a,2}(2,1) = 1/2 - \epsilon, \ \pi^{a,2}(1,2) = \pi^{a,2}(2,2) = 1/2 + \epsilon, \quad a = 1, \ldots, N - 1.$$

We consider this economy over two time horizons. In the first case, the economy starts off with identical beliefs in a steady state. Beliefs change and we report average moments over 100 periods (25 years). In the second case, we consider the long run and report average moments over 10000 periods.

For comparison, Lettau and Uhlig (2002) report that the quarterly standard deviation of returns of S&P-500 stocks in post-war US data is about 7.5 percent. It is well known in the economic literature that parsimonious economic models fail to match most empirical quantities on asset markets. This failure has given rise to many puzzles such as, for example, the equity premium puzzle, the Sharpe ratio puzzle, the return volatility puzzle, see again Lettau and Uhlig (2002).

4.1.1 The short run

We assume that the economy is initially in its deterministic steady state which we denote by $\bar{\kappa}$. One possible justification for this assumption is that beliefs of agents were identical for a long time and we want to consider the effects of a change in beliefs. Table 2 reports $Std_{\bar{\kappa}}^{100}(R^f)$ and $Std_{\bar{\kappa}}^{100}(R^{tree})$ for different combinations of the discount factor δ the belief deviation ϵ. All entries are in percent. Table 2 shows that with relatively small differences

$\delta \backslash \epsilon$	0.1		0.2		0.3	
0.99	0.10	1.53	0.26	4.71	0.37	7.38
0.996	0.10	2.05	0.28	6.54	0.42	10.70
1.00	0.11	2.62	0.29	8.39	0.46	13.67
1.005	0.11	3.16	0.31	10.88	0.52	19.15

Table 2: Second moments (in %) – persistent differences in beliefs

in beliefs volatility of stock returns can be large in the short-run. The effect becomes larger with larger values of δ. In light of the theoretical findings of the previous section these results are easy to interpret. Large values of δ lead to large fluctuations in asset prices as wealth gets redistributed across generations. Large values of ϵ, i.e. large differences in beliefs, lead to large fluctuations in the wealth distribution. This last effect is perhaps not obvious. Naively, one might conjecture that each type only trades with the other type of his generation and that therefore the differences in beliefs we consider here have no effect on the wealth distribution across generations. But this is obviously false: Since there are no borrowing constraints, the young are willing to take much larger bets than the old. So if the same shock repeats itself several times agents of all ages that believe that this shock is more likely become relatively rich, but the young disproportionally so than the old. If differences in beliefs are large, the old are on average poorer than with identical beliefs since somewhere along their life-cycle they "gamble" away not only their current cash-at-hand but also a substantial fraction of future incomes. We return to this issue below when we discuss the market price of risk.

4.1.2 The long run

Table 3 reports the long-run measures $Std_{\bar{\kappa}}^{10000}(R^f)$ and $Std_{\bar{\kappa}}^{10000}(R^{tree})$. All entries are in percent. As in the short run, stock-price volatility is generally large and increases with δ. Somewhat surprisingly the volatility is not always increasing in ϵ. If differences in beliefs are large, $\epsilon = 0.3$, the wealth is concentrated among the young and it happens very rarely that the old become wealthy. Average interest rates actually become negative and the volatility is smaller than for the intermediate case, $\epsilon = 0.2$.

Compared to the short run, it is interesting to note that volatility is much larger for

$\delta\backslash\epsilon$	0.1		0.2		0.3	
0.99	0.73	3.18	1.00	4.92	0.66	4.49
0.996	0.82	4.26	1.08	6.96	0.79	6.91
1.00	0.88	5.18	1.22	8.90	0.71	8.98
1.005	1.03	6.47	1.47	11.93	0.99	12.55

Table 3: Second moments (in %) – persistent differences in beliefs

$\epsilon = 0.1$ while it is smaller for larger values of ϵ. The results in the short run are mainly driven by the fact that initially the economy is in its deterministic steady states and the average wealth distribution with heterogenous beliefs looks quite different than in this steady state. This transition dominates the results in the short run. We return to this point in the discussion of temporary disagreement in Section 4.3.

Note that the results are robust in δ. Even for relatively low values such as $\delta = 0.99$ the effects are still sizable.

4.1.3 The risk-free rate and the market price of risk

Due to the large stock-price volatility generated by the model, the equity premium is sizable. However, the Sharpe ratio (or market price of risk) is still unrealistically small. It is interesting to report how it varies across specifications of the model. We focus on the case $\delta = 1$. In the data, the Sharpe ratio for quarterly returns of the S&P 500 index lies around 0.25. Table 4 reports the average (quarterly) interest rate (in percent) and the Sharpe ratio.

Measure $\backslash\epsilon$	0.1	0.2	0.3
$E_{\bar{\kappa}}^{100}(R^f)$	0.857	0.705	0.433
$E_{\bar{\kappa}}^{10000}(R^f)$	0.579	0.066	-0.230
$\frac{E_{\bar{\kappa}}^{100}(R^{tree}-R^f)}{Std_{\bar{\kappa}}^{100}(R^{tree})}$	0.064	0.103	0.140
$\frac{E_{\bar{\kappa}}^{10000}(R^{tree}-R^f)}{Std_{\bar{\kappa}}^{10000}(R^{tree})}$	0.069	0.099	0.112

Table 4: Interest rates (in %) and Sharpe ratios – persistent differences in beliefs

The Sharpe ratio becomes quite large when differences in beliefs are large (although not surprisingly, since risk aversion is low and markets are complete it obviously never gets close to the empirically observed values). The differences in beliefs and the resulting changes in the wealth distribution not only affect the volatility of the stock price but also have large effects on the Sharpe ratio. Recall that Proposition 1 holds in this economy – for a given wealth distribution, differences in beliefs have no effect on asset prices. The entire mechanism only works through changes in the wealth distribution. So how is it possible

that the Sharpe ratio increases substantially as beliefs become more diverse? It turns out that in this economy a large fraction of the stock is held by the young. For them, the price of the stock decreases precisely when they become relatively poor, i.e. when they lose wealth to the old. Therefore the stock is disproportionally more risky for them than for the other agents in the economy. They require a premium to hold it which is reflected in the Sharpe ratio.

Figure 1 shows the evolution of the stock price in the first 1000 periods of the simulation for the case $\delta = 1$ and $\epsilon = 0.2$. In the figure, the dotted line shows the average stock price over 500 simulations, while the solid line shows the stock price in one (typical) simulation. The figure clarifies our verbal argument above; on average, the stock price increases substantially in the initial periods. This effect clearly leads to high stock returns in this initial phase, but also shows that the typical stockholder is now much younger than in the economy with identical beliefs. For this typical stockholder, the stock is very risky. This becomes clear from the one sample path we show. Stock-price volatility is obviously very large.

[Figure 1 HERE]

4.2 Crowding out of incorrect beliefs?

We now assume that there are three types, one of them holding the correct beliefs, while the other types' beliefs are as in the previous specification. Denote by λ the fraction of type 1 agents that have the correct beliefs, i.e. who think the shocks are equiprobable. We focus on the case $\delta = 1$ and vary ϵ, i.e. the beliefs of type 2 and 3 agents. (Both types make up $(1 - \lambda)/2$ of the population.) Table 5 reports the long-run measures $Std_{\bar{\kappa}}^{10000}(R^f)$ and $Std_{\bar{\kappa}}^{10000}(R^{tree})$. The table shows that when only a small fraction of agents has correct

$\lambda \backslash \epsilon$	0.1		0.2		0.3	
0.1	0.79	5.02	1.20	6.94	1.30	7.23
0.25	0.71	4.93	0.80	5.32	0.97	5.65
0.5	0.36	3.09	0.43	3.59	0.48	4.13
0.75	0.17	2.12	0.32	2.36	0.23	2.55
0.9	0.09	1.01	0.16	1.38	0.17	1.50

Table 5: Second moments (in %) – persistent differences in beliefs

beliefs the stock-price volatility is quite similar to our first specification. As the fraction of agents with the correct beliefs becomes larger, the impact of the two agents with opposite beliefs vanishes. For the case where only 10 percent of all agents have incorrect beliefs, the volatility of stock-returns drops to 1.38 for $\epsilon = 0.2$. However, this number is a little

misleading. In Figure 2, we plot a typical time series of 10000 periods for the stock price. While average volatility is very small, large swings in stock prices are still possible. As the figure shows it is not atypical that in 10000 periods the stock price can move by a factor of more than 2.

[Figure 2 HERE]

The average interest rates and Sharpe ratios are quite far off from empirically observed values. The quarterly interest rate is about 1 percent and the Sharpe ratios are negative, although they are very small. The reason for this is that as opposed to the previous case, now the old hold a large fraction of the stock. This leads to a high average interest rate and to negative Sharpe ratios: When the old become poor, the young must become rich, stock prices must go up and stock returns are high. The stocks are a good hedge for the old against losing bets to the young.

4.3   Temporary disagreements

Finally we consider a specification of our OLG economy in which agents agree most of the time but in which regime-switches lead to temporary disagreement. The disagreement is as before: One agent holds the correct beliefs and the other two agents disagree symmetrically. This disagreement is temporary and the economy returns to the agreement state with positive probability.

Concretely, we assume that there are three shocks $s = 1, 2, 3$. The true law of motion is as follows,

$$\Pi = \begin{bmatrix} 0.9 & 0.05 & 0.05 \\ 0.2 & 0.4 & 0.4 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

In shock 1 agents all agree and have the true probabilities. As before, agents of type 1 have correct beliefs in all states, i.e. $\pi^{a,1} = \Pi$ for all $a = 1, \ldots, N-1$. In shocks 2 and 3 we have for $a = 1, \ldots, N-1$,

$$\pi^{a,h}(2, 1) = \pi^{a,h}(3, 1) = 0.2, \quad h = 2, 3$$

but

$$\pi^{a,2}(2, 2) = \pi^{a,2}(3, 2) = 0.4 + \epsilon \quad \text{and} \quad \pi^{a,3}(2, 2) = \pi^{a,3}(3, 2) = 0.4 - \epsilon.$$

As before, we focus on $\delta = 1$ and vary ϵ and the fraction λ of agents with the correct beliefs, λ. Table 6 reports the long-run measures $Std_{\bar{\kappa}}^{10000}(R^f)$ and $Std_{\bar{\kappa}}^{10000}(R^{tree})$. The table shows that although disagreement is temporary and fundamentals are deterministic there can be large volatility in the stock return when the agents with the correct beliefs are not in the majority. The steady-state distribution of the underlying Markov chain of exogenous

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$\lambda \backslash \epsilon$	0.1		0.2		0.3	
0.5	0.21	1.03	0.37	2.09	0.52	3.17
0.1	0.35	1.75	0.68	3.77	1.21	5.72

Table 6: Second moments (in %) – temporary disagreements

states is $(2/3, 1/6, 1/6)$. So there is disagreement only one-third of the time. Nevertheless if disagreement is substantial quarterly stock return volatility exceeds 5.7 percent.

The average quarterly interest rate is about 0.9 percent to 1 percent and therefore much higher than in the data, the Sharpe ratio is positive but close to zero. Higher values of the discount factor δ obviously lead to more realistic first moments. However, the purpose of the exercise is not to match the moments in the data but rather to show that changes in the wealth distribution caused by differences in beliefs are a quantitatively important source of price volatility in this model.

5   Aggregate uncertainty and the wealth distribution

Up to this point of our analysis we only examined specifications of the model without uncertainty in endowments and dividends. As a consequence, asset price volatility was always zero in the presence of identical beliefs. In this section we analyze asset price volatility for economies with aggregate uncertainty but identical beliefs.

Previous research revealed that in many specifications of the overlapping generations model with aggregate uncertainty the wealth distribution changes very little in equilibrium if beliefs are identical (see e.g. Rios-Rull (1996) and Storesletten et al. (2007)). In our model we can provide a theoretically precise description of these findings. Under the assumptions of the now following Theorem 3, the economy has a stochastic steady state, that is, there exist initial conditions κ such that the equilibrium exhibits no fluctuations in the wealth distribution and prices and choices depend on the exogenous shock alone.

THEOREM 3 *Consider an economy where all agents $a = 1, \ldots, N$, $h = 1, \ldots, H$, have identical and correct beliefs, $\pi^{a,h} = \Pi$. Then, under either of the following two assumptions, there exist initial conditions κ such that in the resulting equilibrium, prices and consumption choices are time invariant functions of the exogenous shock alone.*

1. *All endowments and dividends are collinear, i.e. for all agents $a = 1, \ldots, N$, $h = 1, \ldots, H$, it holds that*

$$\frac{e^{a,h}(s)}{e^{a,h}(s')} = \frac{d(s)}{d(s')} \quad \text{for all} \ \ s, s' = 1, \ldots, S.$$

2. *Shocks are i.i.d., i.e. for all shocks s', $\Pi(s, s')$ is independent of s, and endowments of all*

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agents of age $a = 1$ are collinear to aggregate endowments, i.e. for all $h = 1, \ldots, H$,

$$\frac{e^{1,h}(s)}{e^{1,h}(s')} = \frac{\omega(s)}{\omega(s')} \quad \text{for all } s, s'.$$

The theorem describes two benchmark cases of our OLG model with aggregate uncertainty for which the wealth distribution remains constant along the equilibrium path and thus does not matter for equilibrium allocations and prices. While we can prove the theorem only for a specific initial condition, we found in many simulations that if the economy starts from other initial conditions then the equilibrium quickly converges to the stochastic steady state with a constant wealth distribution.

Commonly applied realistic calibrations of asset pricing models deviate from the assumptions of Theorem 3 in at least two directions. Either labor endowments are assumed to be safe or shocks to labor endowments are assumed to be independent of shocks to dividends. The question arises whether such calibrations of our OLG model lead to substantially different equilibrium predictions. They do not. The quantitative effects in calibrated economies are tiny, fluctuations in the wealth distribution have little impact on stock price volatility.

For an illustration of these results we consider an economy with both endowment and dividend shocks. There is only one type of agent per generation, so $H = 1$. To emphasize the point, we deliberately consider very large shocks; for smaller shocks, the effects are obviously much smaller. Specifically, let

$$d(1) = d(2) = 0.9, \ d(3) = d(4) = 1.1 \ \text{ and } \ e^a(1) = e^a(3) = 0.9e^a, \ e^a(2) = e^a(4) = 1.1e^a$$

where the labor endowments e^a, $a = 1, \ldots, N$, are calibrated as in Section 4 above. For simplicity, let beliefs be such that $\Pi(s, s') = 1/4$ for all s, s'.

If the wealth distribution did matter, prices conditional on a given exogenous shock would fluctuate substantially. For each shock $s = 1, \ldots, 4$, we therefore compute the standard deviation of the stock price conditional on this shock as well as the average stock price. Table 7 reports average prices and (conditional) standard deviations for the 4 shocks.

	s=1	s=2	s=3	s=4
Avg. price	93.71	100.66	107.59	114.54
Std. dev.	0.0018	0.0020	0.0021	0.0023

Table 7: Large exogenous shocks – small price volatility

Obviously the movements in the wealth distribution must be tiny. This result is in large part due to market completeness. Theorem 3 provides an excellent approximation because the only deviation from the assumptions of the theorem is the lack of collinearity of aggregate endowments and the endowments of the agent of age 1. While the difference is large for that agent, he only makes up a very small part of the whole economy.

6 Conclusion

We consider a simple model of asset prices in an exchange economy with overlapping generations. Our model is both canonical and parsimonious – financial markets are complete and there are no restrictions on asset trades. We show that with identical beliefs the asset pricing implications of this model are similar to the implications of a standard Lucas (1978) model with a representative agents. This finding is in line with the previous literature. However, differences in opinion, even if they are small and unrelated to fundamentals in the economy, can lead to completely different asset pricing implications. In particular, the model generates high excess volatility.

Appendix

A   Proofs

For the proofs of Theorem 1, Proposition 1 and Corollary 1 it is useful to consider the Arrow-Debreu equilibrium of the OLG economy. In finite Arrow-Debreu economies, the assumption of Cobb-Douglas utility implies that the equilibrium is unique and that equilibrium prices are the solution of a linear system of equations. These insights carry over to the Arrow-Debreu equilibrium of our OLG model although the technical details are more complicated. Since it follows from Santos and Woodford (1997) that there cannot be bubbles in our sequential equilibrium, there is a one-to-one correspondence between the Arrow-Debreu equilibria and the sequential equilibria.

We first define the endowments of each agent appearing in the OLG economy at each node of the event tree. Recall that we identify individuals (born into the economy) by the date-event of their birth, $\sigma \in \Sigma$, and their type $h = 1, \ldots, H$. For each such agent define his endowment $\omega^{\sigma,h}(s^t) = e^{\sigma,h}(s^t)$ for all nodes $s^t \in \Sigma$. Of course, the definition of e implies that an agent's endowment is zero at all nodes at which he is not alive. Recall that we denote individuals who are born before $t = 0$ by (s^{1-a}, h) for $a = 2, \ldots, N$ and $h = 1, \ldots, H$. These agents' endowments include the dividends of the Lucas-tree over the whole event tree and are thus given by $\omega^{s^{1-a},h}(s^t) = e^{s^{1-a},h}(s^t) + \phi^{s^{1-a},h}d(s^t)$ for all nodes $s^t \in \Sigma$. The aggregate endowment in the economy is

$$\omega(s^t) = \omega(s_t) = \sum_{\sigma \in \Sigma} \sum_{h=1}^{H} \omega^{\sigma,h}(s^t) + \sum_{a=2}^{N} \sum_{h=1}^{H} \omega^{-a+1,h}(s^t).$$

Note that the aggregate endowment only depends on the current shock s_t. Denote the price for the consumption good at each node $s^t \in \Sigma$ by $\rho(s^t)$ with the normalization $\rho(s_0) = 1$.

The Arrow-Debreu equilibrium is defined as usual by prices $(\rho(\sigma))_{\sigma \in \Sigma}$ and consumption allocations $c^{\sigma,h}$ such that markets clear and agents maximize utility.

Observe that a finite number of agents, namely those born before $t = 0$, hold a non-negligible fraction of the aggregate endowment at all nodes. This fact ensures the existence of a competitive equilibrium, see Geanakoplos and Polemarchakis (1991, Theorem 2). Moreover, in every equilibrium the value of the aggregate endowment must be finite. Our assumption of log utility implies that excess demand functions satisfy the gross substitute property. As a consequence of these last two properties the Arrow-Debreu equilibrium is unique, see Kehoe et al. (1991, Theorem A).

Building on the existence of a unique Arrow-Debreu equilibrium we can now prove Theorem 1.

Proof of Theorem 1. We first determine the income of all agents in the economy. The income of agent (s^{1-a}, h) who is born before $t = 0$ is given by

$$I^{s^{1-a},h} = \phi^{s^{1-a},h} \sum_{\sigma \in \Sigma} \rho(\sigma) d(\sigma) + \sum_{n=a}^{N} \sum_{s^{n-a} \succeq s_0} \rho(s^{n-a}) e^{n,h}(s_{n-a})$$

for $a = 2, \ldots, N$ and $h = 1, \ldots, H$. Analogously to our cash-at-hand definition for the sequential equilibrium we can define cash-at-hand for the Arrow-Debreu equilibrium as

$$\kappa^{s^{1-a},h} = \phi^{s^{1-a},h} \sum_{\sigma \in \Sigma} \rho(\sigma) d(\sigma)$$

for $a = 2, \ldots, N-1$ and $h = 1, \ldots, H$. With this expression we can rewrite agents' income as follows,

$$I^{s^{1-a},h} = \kappa^{s^{1-a},h} + \sum_{n=a}^{N} \sum_{s^{n-a} \succeq s_0} \rho(s^{n-a}) e^{n,h}(s_{n-a}) \quad \text{for } a = 2, \ldots, N-1, \tag{10}$$

$$\sum_{h=1}^{H} I^{s^{1-N},h} = \sum_{\sigma \in \Sigma} \rho(\sigma) d(\sigma) - \left(\sum_{a=2}^{N-1} \sum_{h=1}^{H} \kappa^{s^{1-a},h} \right) + \rho(s_0) \sum_{h=1}^{H} e^{N,h}(s_0). \tag{11}$$

The income of agent (σ, h) entering the economy at node $\sigma = s^t \in \Sigma$ is given by

$$I^{\sigma,h} = \sum_{a=1}^{N} \sum_{s^{t+a-1} \succeq \sigma} \rho(s^{t+a-1}) e^{a,h}(s_{t+a-1}). \tag{12}$$

Next we derive the linear system of excess demand equations that determines equilibrium prices. This system is analog to the system in finite economies with Cobb-Douglas utilities with the exception that it has infinitely many equations and unknowns. For each agent s^t, h and age $a = 1, \ldots, N$ define the weight

$$\xi^{s^t,h}(s^{t+a-1}) = \frac{\delta^{a-1} \pi^{s^t,h}(s^{t+a-1}|s^t)}{\sum_{j=0}^{N-1} \delta^j}. \tag{13}$$

For the agents (s^{1-a}, h) who are present before $t = 0$ the coefficients $\xi^{s^{1-a},h}(s^t)$ are given by

$$\xi^{s^{1-a},h}(s^t) = \frac{\delta^t \pi^{s^{1-a},h}(s^t|s_0)}{\sum_{j=0}^{N-a} \delta^j}, \quad t = 0, \ldots, N - a,$$

for $a = 2, \ldots, N$ and s^t with $t = 0, \ldots, N - a$. Observe that in the case $a = N$,

$$\xi^{s^{1-N},h}(s_0) = 1 \text{ for all } h = 1, \ldots, H. \tag{14}$$

The Arrow-Debreu prices, normalized such that $\rho(s_0) = 1$, are now the unique solution to the following linear system of equations.

$$\sum_{a=1}^{N} \sum_{h=1}^{H} \xi^{s^{1-a},h}(s_0) \, I^{s^{1-a},h} = \omega(s_0) \tag{15}$$

$$\sum_{a=1}^{N} \sum_{h=1}^{H} \xi^{s^{t+1-a},h}(s^t) \, I^{s^{t+1-a},h} = \rho(s^t)\omega(s^t), \quad \text{for all } s^t \succ s_0 \tag{16}$$

Observe that we can eliminate the income variables of the agents who are of age N at $t = 0$ from Equation (15) since they both appear with a weight of 1, see Condition (14), by the right-hand side of Equation (11). These two income variables do not appear in Equations (16). Moreover, we can replace the incomes of all other agents by the corresponding expressions from Equations (10) and (12). A close inspection of the resulting system of infinitely many equations and unknowns reveals that all equations are linear in the cash-at-hand positions $\kappa^{s^{1-a},h}$ for $a = 2, \ldots, N - 1$, $h = 1, \ldots, H$, and the unknown Arrow-Debreu prices $\rho(s^t)$ for $s^t \succ s_0$. Therefore, the Arrow-Debreu prices are a linear (affine) function of the initial positions $\kappa^{s^{1-a},h}$. As a result the incomes and thus the consumption allocations of all agents are linear functions of the initial conditions. In particular, the individual consumption allocations at s_0 are linear in the cash-at-hand positions. The same must be true for the price of the Lucas-tree, $q(s_0) = \sum_{s^t \succ s_0} \rho(s^t)d(s^t)$, and the price of a riskless one-period bond, $1/R_f(s_0) = \sum_{s_1 \in \mathcal{S}} \rho(s_1)$.

Since the Arrow-Debreu equilibrium is unique for all initial conditions, the sequential equilibrium that implements the Arrow-Debreu outcome must be recursive with the state consisting of the exogenous shock $s \in \mathcal{S}$ and the beginning of period cash-at-hand across agents, see Kubler and Schmedders (2002).

Finally, note that all coefficients in the pricing and consumption functions, α, β, γ, must be non-negative: If one of the coefficients were negative, we could find initial conditions for a modified economy where the agents of age N at $t = 0$ have arbitrarily large endowments and would obtain negative prices or negative individual consumptions. $\square$

Proof of Proposition 1. For a model without uncertainty we can simplify the linear system of equations (15)–(16) that determines the Arrow-Debreu prices. Without uncertainty we can identify an agent by the date of his birth, t, and his type, h. The weights (13)

aggregate to

$$\hat{\xi}^{t,h}(t+a-1) = \frac{\delta^{a-1}}{\sum_{j=0}^{N-1} \delta^j}$$

for $a = 1, \ldots, N$. Similarly, expression (12) reduces to

$$\hat{I}^{t,h} = \sum_{a=1}^{N} \hat{\rho}(t+a-1)e^{a,h}$$

where $(\hat{\rho}(t))$ denotes the sequence of Arrow-Debreu prices. For the agents who are present before time $t = 0$ the weights are

$$\hat{\xi}^{1-a,h}(t) = \frac{\delta^t}{\sum_{j=0}^{N-a} \delta^j}, \quad t = 0, \ldots, N-a,$$

for $a = 2, \ldots, N$. These agents' income is

$$\hat{I}^{1-a,h} = \kappa^{1-a,h} + \sum_{j=a}^{N} \hat{\rho}(j-a)e^{j,h}.$$

In an economy with deterministic fundamentals but several states we have for any t that

$$\sum_{s^t \in \Sigma} \xi^{s^{t-a+1},h}(s^t)I^{s^{t-a+1},h} = \hat{\xi}^{t-a+1,h}(t)\hat{I}^{t-a+1,h}.$$

with $\sum_{s^t \in \Sigma} \rho(s^t) = \hat{\rho}(t)$. (Note that we must add over all possible date-events s^t in the event tree at time t.) In the economy without uncertainty, the Arrow-Debreu prices are therefore determined by the simplified linear system

$$\sum_{a=1}^{N} \sum_{h=1}^{H} \hat{\xi}^{1-a,h}(0) \ I^{1-a,h} = \omega \tag{17}$$

$$\sum_{a=1}^{N} \sum_{h=1}^{H} \hat{\xi}^{t+1-a,h}(t) \ I^{t+1-a,h} = \hat{\rho}(t)\omega, \quad \text{for all } t = 1, 2, \ldots \tag{18}$$

where $\omega \equiv \omega(t)$, $t = 0, 1, \ldots$, denotes the constant aggregate endowment. Observe that this system of equation does not depend on agents' beliefs.

The prices $\rho(s^t)$ solve the general system of linear equations (15) and (16) if and only if the prices $\hat{\rho}(t)$ solve the specialized system (17) and (18). Since the general system has a unique solution so does the specialized system. This solution does not depend on agents' beliefs and thus the same must be true for the price of the risk-free bond and the price of the Lucas-tree.

The consumption of an agent of age a and type h alive at $t = 0$ is given by $\hat{\xi}^{1-a,h}(0) \ I^{1-a,h}$. Clearly this is the same linear function of cash-at-hand for all beliefs. $\square$

Proof of Corollary 1. We rewrite Equation (18) for $t = 1$ and the special case $e^{a,h} = 0$ for $a > 1$ and $e^{1,1} = 1$. For simplicity we write this equation recursively and denote by κ^a the cash-at-hand of agents of age a (at $t = 0$). We obtain

$$\frac{1}{\sum_{j=0}^{N-1} \delta^j} \hat{\rho}(1) + \frac{\delta}{\sum_{j=0}^{N-1} \delta^j} + \frac{\delta}{\sum_{j=0}^{N-2} \delta^j} \kappa^2 + \ldots + \frac{\delta}{1+\delta} \kappa^{N-1} = \hat{\rho}(1)(1+d).$$

Note that the income of the agent of age 2 at $t = 1$ is $\hat{I}^0 = \hat{\rho}(0)e^1 = 1$, while the income of the agent of age 1 is $\hat{\rho}(1)$. Solving for the bond price, $\hat{\rho}(1)$, then gives the desired result. $\square$

Derivation of Equation (9). Following Equation (7) the price of the riskless bond is

$$
\begin{aligned}
1/R^f(s^t) &= \gamma_1 + \sum_{a=2}^{N-1} \gamma_a \phi^{a-1}(s^t) \frac{\frac{N-1}{N} + d}{1 - \sum_{a=2}^{N-1} \beta_a \phi^{a-1}(s^t)} \\
&= \frac{1}{N(d+1)-1} + \left(\frac{N-1}{N} + d\right) \frac{\sum_{a=2}^{N-1} \gamma_a \phi^{a-1}(s^t)}{1 - \sum_{a=2}^{N-1} \beta_a \phi^{a-1}(s^t)} \\
&= \frac{1}{N(d+1)-1} + \\
&\qquad \frac{N(d+1)-1}{N} \left(\frac{N}{N(d+1)-1}\right) \frac{\sum_{a=2}^{N-1} \left(\frac{1}{N-a+1}\right) \phi^{a-1}(s^t)}{1 - \sum_{a=2}^{N-1} \left(1 - \frac{1}{N-a+1}\right) \phi^{a-1}(s^t)} \\
&= \frac{1}{N(d+1)-1} + \frac{\sum_{a=2}^{N-1} \left(\frac{1}{N-a+1}\right) \phi^{a-1}(s^t)}{1 - \sum_{a=2}^{N-1} \left(1 - \frac{1}{N-a+1}\right) \phi^{a-1}(s^t)}. \qquad \square
\end{aligned}
$$

Proof of Theorem 2. We construct an economy with 3 shocks $s = 1, 2, 3$, that exhibits the desired volatility. It suffices to consider an economy with one type per generation, $H = 1$. Let $\delta = 1$ and set $e^1 = 1$ and $e^a = 0$ for $a = 2, \ldots, N$. Denote by $\bar{p}$ the (hypothetical) price of the tree if all wealth is held by the agent of generation $a = 2$ and denote by $\underline{p}$ the price of the tree if all wealth is held by the agent of generation $N - 2$. Given a bound $\bar{v}$, the discussion in Section 3.1 implies that for all sufficiently large N $\bar{p} - \underline{p} > 2\bar{v} + 1$.

For the described specification of the OLG model, the proof of Proposition 1 (or, in fact, Huffman (1987)) implies that the consumption function of an agent of age a is independent of beliefs and just depends on his cash-at-hand. For $\delta = 1$ this function is given by

$$c^a(\kappa^a) = \frac{1}{N-a+1} \kappa^a \quad \text{for} \quad a = 2, \ldots, N.$$

Agents of age 1 always consume $1/N$ and the aggregate consumption of all other agents is $\frac{N-1}{N} + d$. For $a \geq 2$ the consumption function is injective, that is, it is a one-to-one mapping between individuals' cash-at-hand and consumption allocation. Thus there exist $\epsilon > 0$ such the equilibrium tree price exceeds $\bar{p} - 0.1$ if and only if $c^2 \geq (1 - \epsilon)(\frac{N-1}{N} + d)$ and this price is below $\underline{p} + 0.1$ if and only if $c^{N-1} \geq (1 - \epsilon)(\frac{N-1}{N} + d)$. We now construct

an economy for which the equilibrium allocations satisfy these properties and choose the "true" probabilities so that the desired volatility is exhibited in equilibrium.

Choose the true law of motion to be $\Pi(s,1) = \Pi(s,2) = 1/2$ and thus $\Pi(s,3) = 0$ for all $s = 1, 2, 3$. All agents' subjective beliefs are i.i.d., that is, we can write π_s^a to denote the subjective probability that the agent of age $a = 1, \ldots, N-1$ attaches to shock s in the next period. Choose $\pi_1^1 < 1$ close to one and define the price of the Arrow security for state 1 in the next period by

$$q_1 = \pi_1^1 \frac{1}{N(\frac{N-1}{N} + d)(1 - \epsilon)}.$$

This is the supporting price that ensures that in the next period the agent of age 2 consumes exactly $(\frac{N-1}{N} + d)(1 - \epsilon)$ if shock 1 occurs. Whenever the equilibrium price for this Arrow security is below this supporting price, the agent consumes more. Analogously, choose $\pi_2^{N-2} < 1$ close to one and define for some lower bound on consumption, $\underline{c}$,

$$q_2(\underline{c}) = \pi_2^{N-2} \underline{c} \frac{1}{(\frac{N-1}{N} + d)(1 - \epsilon)}.$$

To achieve market clearing we now have to show that there exist probabilities such that for these prices all other agents choose consumption below $(\frac{N-1}{N} + d)\epsilon/(N-2)$ and that consumption of agents of age $N-2$ is bounded below by $\underline{c} > 0$. Define $\pi_1^a = \pi_2^a = \zeta$ for all $a = 2, \ldots, N-3$ as well as $\pi_2^1 = \pi_1^{N-2} = \zeta$. Clearly we can choose $\zeta \leq \epsilon$ small enough to ensure that the agent of age 3 consumes below $(\frac{N-1}{N} + d)\epsilon/(N-2)$ even if the previous state was $s = 1$ and he consumed $(1 - \epsilon)(\frac{N-1}{N} + d)$ in the previous period when he was of age 2. This choice of beliefs also ensures that all other agents consume below $(\frac{N-1}{N} + d)\epsilon/(N-2)$ on the equilibrium path. In fact, it follows from the first order conditions that it suffices to choose

$$\zeta = \epsilon/(N-2) \min\{q_1, q_2(\underline{c})\}.$$

Iterating on the first order conditions then yields a lower bound $\underline{c}$ on the consumption of agents of age $N-2$, namely

$$\underline{c} \geq \frac{\epsilon^{N-3}}{(N-2)^{N-3}} \frac{1}{N}.$$

Finally, we choose $\pi_1^{N-1} = \pi_2^{N-1} > 0$ sufficiently small to ensure that via equation (9) the bond price never falls much below its maximal value and thus never varies by more than $\underline{v}$. This completes the proof of the theorem. $\square$

Proof of Theorem 3. We prove the theorem by a "guess and verify" approach. We guess that consumption allocations are collinear and then derive values for all endogenous variables that satisfy the equilibrium equations.

Suppose that consumption only depends on the shock and that individual consumption allocations are given by $c_s^{a,h} = \nu_{a,h}\omega_s$ with $\nu_{a,h} > 0$ and $\sum_{a=1}^N \sum_{h=1}^H \nu_{a,h} = 1$ for all states

$s = 1, 2, \ldots, S$. Substituting these consumption allocations into the Euler equations yields the prices of the Arrow securities,

$$q_{ss'} = \delta \Pi(s, s') \frac{\nu_{a,h}\omega_s}{\nu_{a+1,h}\omega_{s'}}, \quad \forall\, a = 1, \ldots, N-1,\ h = 1, \ldots, H. \tag{19}$$

The asset prices $q_{ss'}$ are obviously independent of the agent and thus we can define a new constant f such that

$$f \equiv \delta \frac{\nu_{a,h}}{\nu_{a+1,h}} \quad \forall\, a = 1, \ldots, N-1,\ h = 1, \ldots, H. \tag{20}$$

We can write an agent's lifetime budget constraint, if he does not initially own shares of the tree, as follows,

$$c_s^{1,h} - e_s^{1,h} + \sum_{s'=1}^{S} q_{ss'} \left(c_{s'}^{2,h} - e_{s'}^{2,h} + \sum_{s''=1}^{S} q_{s's''} \left(c_{s''}^{3,h} - e_{s''}^{3,h} + \sum_{s'''=1}^{S} q_{s''s'''} \left(c_{s'''}^{4,h} - e_{s'''}^{4,h} + \ldots \right) \right) \right) = 0.$$

Case 1. Individual labor endowments are collinear, that is, for each agent (a, h), there is a weight $\eta_{a,h}$ such that his endowments are given by $e_s^{a,h} = \eta_{a,h}\omega_s$. Note that the weights $\eta_{a,h} \geq 0$ do not sum to 1, $\sum_{a=1}^{N}\sum_{h=1}^{H} \eta_{a,h} < 1$, since the social endowment ω_s includes dividends d_s.

In this case

$$c_s^{1,h} - e_s^{1,h} = (\nu_{1,h} - \eta_{1,h})\,\omega_s$$

and using (19) and (20) also

$$\begin{aligned}
\sum_{s'=1}^{S} q_{ss'}(c_{s'}^{2,h} - e_{s'}^{2,h}) &= \sum_{s'=1}^{S} \Pi(s, s')f \frac{\omega_s}{\omega_{s'}}(\nu_{2,h} - \eta_{2,h})\,\omega_{s'} \\
&= f(\nu_{2,h} - \eta_{2,h})\,\omega_s.
\end{aligned}$$

Similarly,

$$\sum_{s'=1}^{S} q_{ss'} \sum_{s''=1}^{S} q_{s's''}(c_{s''}^{3,h} - \omega_{s''}^{3,h}) = f^2(\nu_{3,h} - \eta_{3,h})\,\omega_s$$

and so on. Thus, the budget constraint of an agent of type h, born at shock s is equivalent to

$$\left(\sum_{a=1}^{N} f^{a-1}(\nu_{a,h} - \eta_{a,h}) \right) \omega_s = 0$$

which in turn is equivalent to

$$\sum_{a=1}^{N} f^{a-1}(\nu_{a,h} - \eta_{a,h}) = 0. \tag{21}$$

The definition of the ratio f implies that $f\nu_{a+1,h} = \delta\nu_{a,h}$ and thus $f^{a-1}\nu_{a,h} = \nu_{1,h}\delta^{a-1}$ and so we obtain for each agent h,

$$\nu_{1,h} \sum_{a=1}^{N} \delta^{a-1} - \sum_{a=1}^{N} f^{a-1}\eta_{a,h} = 0.$$

These H equations together with the market-clearing condition

$$\sum_{a=1}^{N}\sum_{h=1}^{H}\nu_{a,h} = \sum_{h=1}^{H}\left(\nu_{1,h}\sum_{a=1}^{N}\frac{\delta^{a-1}}{f^{a-1}}\right) = 1$$

yields a system of $H+1$ equations in the $H+1$ unknowns f and $\nu_{1,h}$, $h = 1,\ldots,H$. Substituting for $\nu_{1,h}$ we obtain a polynomial equation in the single unknown f,

$$f^{N-1}\sum_{a=1}^{N}\delta^{a-1} - \left(\sum_{a=1}^{N}f^{a-1}\sum_{h}\eta_{a,h}\right)\left(\sum_{a=1}^{N}\left(f^{N-a}\delta^{a-1}\right)\right) = 0. \tag{22}$$

Observe that the polynomial on the left-hand side is of the form $g(f) = \sum_{k=0}^{2N-2} r_k f^k$ with coefficients r_k satisfying $r_k < 0$ for $k \neq N-1$ and $r_{N-1} > 0$. The classical Sign Rule of Descartes now implies that equation (22) can have at most two positive solutions. Moreover, $g(0) = r_0 < 0$ and $g(f) \to -\infty$ as $f \to \infty$. And since $g(1) > 0$ the polynomial g has two distinct positive roots, one less than 1 and a second larger than 1.

There are no bubbles in this OLG economy, see Santos and Woodford (1997). Moreover, the stationarity of the prices of the Arrow securities implies that the tree price is also stationary, i.e. $p(s^t) = p_{s_t}$. Agents' Euler equations then require

$$p_s = \sum_{s'} q_{ss'}\left(p_{s'} + d_{s'}\right) \qquad \text{for all} \quad s = 1,\ldots,S,$$

or, equivalently,

$$P = Q\left(P + d\right) \tag{23}$$

where P (d) denotes the S-vector of tree prices (dividends) and Q denotes the $(S \times S)$-matrix of prices of Arrow securities. The matrix Q is the element-wise (Hadamard) product of the rank-one positive matrix Ω with elements $\Omega_{ss'} = \omega_s/\omega_{s'}$ and the matrix $f\Pi$ with largest eigenvalue f. Thus the matrix Q has also largest eigenvalue f. But for the solution $f > 1$ of (22) Equation (23) does not yield a finite solution for P. The equation delivers only for $f < 1$ a finite price vector, namely $P = [I - Q]^{-1} d$. Therefore, the solution $f < 1$ of Equation (22) yields the unique equilibrium. In this equilibrium consumption allocations are collinear. This completes the proof of Case 1.

Case 2. Only the individual labor endowments of cohort 1 are assumed to be collinear with the social endowment, $e_s^{1,h} = \eta_{1,h}\omega_s$. Beliefs are i.i.d. so we can write $\Pi_{s'} \equiv \Pi(s,s')$ for all $s = 1, 2, \ldots, S$. As before we have for cohort 0,

$$c_s^{1,h} - e_s^{1,h} = \left(\nu_{1,h} - \eta_{1,h}\right)\omega_s.$$

Now using (19) and (20) we obtain

$$\sum_{s'=1}^{S} q_{ss'}(c_{s'}^{2,h} - e_{s'}^{2,h}) = \sum_{s'=1}^{S}\Pi_{s'}f\frac{\omega_s}{\omega_{s'}}\left(\nu_{2,h}\omega_{s'} - e_{s'}^{2,h}\right)$$

$$= f\omega_s\left(\nu_{2,h} - \sum_{s'=1}^{S}\Pi_{s'}\frac{e_{s'}^{2,h}}{\omega_{s'}}\right).$$

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Similarly,

$$\sum_{s'=1}^{S} q_{ss'} \sum_{s''=1}^{S} q_{s's''}(c_{s''}^{3,h} - \omega_{s''}^{3,h}) = f^2 \omega_s \left(\nu_{3,h} - \sum_{s''=1}^{S} \Pi_{s''} \frac{e_{s''}^{3,h}}{\omega_{s''}} \right)$$

and so on. Thus, the budget constraint is equivalent to

$$\sum_{a=1}^{N} f^{a-1} \nu_{a,h} - \eta_{1,h} - \sum_{a=2}^{N} f^{a-1} \left(\sum_{s^{(i)}=1}^{S} \Pi_{s^{(i)}} \frac{e_{s^{(i)}}^{a,h}}{\omega_{s^{(i)}}} \right) = 0$$

where $s' = s^{(1)}$, $s'' = s^{(2)}$, and so on. Following the same steps as in Case 1 leads to the single polynomial equation

$$f^{N-1} \sum_{i=0}^{N-1} \delta^i - \left(\eta_0 + \sum_{i=1}^{N-1} f^i \left(\sum_{s^{(i)}=1}^{S} \Pi_{s^{(i)}} \frac{e_{s^{(i)}}^i}{\omega_{s^{(i)}}} \right) \right) \left(\sum_{i=0}^{N-1} f^{n-i} \delta^i \right) = 0. \qquad (24)$$

in the single unknown f. As in Case 1 the left-hand side of this equation is a polynomial $g(f)$ with two sign changes and $g(0) < 0$, $g(1) > 0$ and $g(f) \to -\infty$ for $f \to \infty$. Thus there are again two solutions but, again, only the solution $f < 1$ leads to a well-defined stock price. This completes the proof of Case 2. $\square$

B   Numerical solution

Recall that in our OLG economy markets are dynamically complete since each date-event s^t has S successor nodes and agents can trade a full set of S Arrow securities at each date-event. Moreover, the equilibrium consumption allocation is unique as the discussion preceding the proof of Theorem 1 in Appendix A demonstrates. But agents' portfolios are not unique in equilibrium since the agents can trade the Lucas-tree in addition to the Arrow securities. Thus, in equilibrium, a continuum of portfolios supports the unique consumption allocation. At each date-event equilibrium portfolios are a one-dimensional subspace of $\mathbb{R}^S$. For the computation of the linear policy and pricing functions we exploit this multiplicity of portfolios supporting the equilibrium by imposing an additional restriction. This condition on portfolios then uniquely determines one point in the one-dimensional subspace.

B.1   Equilibrium equations

The additional restriction forces the agent of age $N-1$ and type 1 to buy the entire Lucas-tree. He holds it for one period and then sells it in the last, Nth, period of his life to the subsequent agent of age $N-1$ and type 1. All agents of ages $a = 1, \ldots, N-2$ and the type $h = 2, \ldots, H$ agents of age $N-1$ are only permitted to trade Arrow securities. This choice of equilibrium portfolio greatly simplifies the beginning-of-period cash-at-hand for all agents except for the one of age $N-1$ and type 1. The respective cash-at-hand positions

at date-event $s^{t+1} = (s^t, s_{t+1})$ are then simply $\kappa^{a+1,h}(s^{t+1}) = \theta^{a,h}_{s_{t+1}}(s^t)$, that is, an agent's cash-at-hand is just his holding of the Arrow security that pays in the current shock s_{t+1}.

With the special choice for the equilibrium portfolios we can now derive a nonlinear system of equations that must hold in equilibrium. Let the current shock be $s \in \mathcal{S}$ and the current cash-at-hand positions be $\kappa^{a,h}$ for $a = 2, \ldots, N-1$, and $h = 1, \ldots, H$.

The first set of equations are the necessary and sufficient first-order optimality conditions for the agents' utility maximization problems. The generic first-order conditions with respect to portfolio holdings of the Arrow securities are of the form

$$-q_{s'}(s^t)u'(c^{a,h}(s^t)) + \delta\pi^{a,h}(s'|s)u'(c^{a+1,h}(s^t, s')) = 0 \quad \text{for } a = 1, \ldots, N-1, \quad (25)$$

where $q_{s'}(s^t)$ denotes the price of the Arrow security with a payoff in shock s' in the next period. Substituting the expressions (1) and (2) into these first-order conditions yields the following equations.

For $a = 1$:

$$-q_{s'}\left(\alpha^{2,h}_{1s'} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\alpha^{2,h}_{jis'}\theta^{j-1,i}_{s'}\right) + \delta\pi^{1,h}(s'|s)\left(e^{1,h}_s - \sum_{s'}q_{s'}\theta^{1,h}_{s'}\right) = 0. \quad (26)$$

For $a = 2, \ldots, N-2$:

$$-q_{s'}\left(\alpha^{a+1,h}_{1s'} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\alpha^{a+1,h}_{jis'}\theta^{j-1,i}_{s'}\right) + \delta\pi^{a,h}(s'|s)\left(e^{a,h}_s + \kappa^{a,h} - \sum_{s'}q_{s'}\theta^{a,h}_{s'}\right) = 0. \quad (27)$$

For $a = N-1$ and $h = 1$:

$$-q_{s'}\left(e^{N,1}_{s'} + \theta^{N-1,1}_{s'} + d_{s'} + \beta_{1s'} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\beta_{jis'}\theta^{j-1,i}_{s'}\right) + \delta\pi^{N-1,1}(s'|s)$$

$$\left(e^{N-1,1}_s + \kappa^{N-1,1} - \sum_{s'}q_{s'}\theta^{N-1,1}_{s'} - \left(\beta_{1s} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\beta_{jis}\kappa^{j,i}\right)\right) = 0. \quad (28)$$

For $a = N-1$ and $h = 2, \ldots, H$:

$$-q_{s'}\left(e^{N,h}_{s'} + \theta^{N-1,h}_{s'}\right) + \delta\pi^{N-1,h}(s'|s)\left(e^{N-1,h}_s + \kappa^{N-1,h} - \sum_{s'}q_{s'}\theta^{N-1,h}_{s'}\right) = 0. \quad (29)$$

Next we have the first-order condition of the agent of age $N-1$ and type 1 with respect to his holding of the Lucas-tree,

$$-p(s^t)u'\left(c^{N-1,1}(s^t)\right) + \delta\sum_{s'=1}^{S}\pi^{N-1,1}(s'|s)\left(u'(c^{N,1}(s^t, s'))\left(d_{s'} + p(s^t, s')\right)\right) = 0.$$

Using the prices of the Arrow securities, see Equation (25), we can write the condition on the price of the Lucas-tree as follows,

$$-\left(\beta_{1s} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\beta_{jis}\kappa^{j,i}\right) + \sum_{s'}q_{s'}\left(d_{s'} + \beta_{1s'} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\beta_{jis'}\theta_{s'}^{j-1,i}\right) = 0. \qquad (30)$$

This equation completes the set of equations derived from agents' first-order conditions.

We have S market-clearing equations.

$$\sum_{a=1}^{N-1}\sum_{h=1}^{H}\theta_{s'}^{a,h} = 0 \quad \text{for } s' = 1, \ldots, S. \qquad (31)$$

The third and last set of equations imposes consistency conditions on the linear consumption functions.

For $a = 2, \ldots, N-2$, $h = 1, \ldots, H$, and the agents of age $N-1$ and type $h = 2, \ldots, H$:

$$\alpha_{1s}^{a,h} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\alpha_{jis}^{a,h}\kappa^{j,i} = e_s^{a,h} + \kappa^{a,h} - \sum_{s'}q_{s'}\theta_{s'}^{a,h} \qquad (32)$$

For the agent of age $N-1$ and type 1

$$\alpha_{1s}^{N-1,1} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\alpha_{jis}^{N-1,1}\kappa^{j,i} = e_s^{N-1,1} + \kappa^{N-1,1} - \sum_{s'}q_{s'}\theta_{s'}^{N-1,1} - \left(\beta_{1s} + \sum_{j=2}^{N-1}\sum_{i=1}^{H}\beta_{jis}\kappa^{j,i}\right) \qquad (33)$$

Equations (26)–(33) must hold for each $s \in \mathcal{S}$ and each initial condition $\kappa^{a,h}$ for $a = 2, \ldots, N-1$ and $h = 1, \ldots, H$. For fixed s and fixed initial condition, Equations (26)–(30) consist of $H(N-1)S + 1$ equations. In addition, there are S market-clearing equations. Finally there are $H(N-2)S$ consistency conditions. For all $s \in \mathcal{S}$ combined there are

$$S\left(H(N-1)S + 1 + S\right) + H(N-2)S$$

equations. Observe that unlike the first-order conditions and market-clearing equations the consistency conditions appear exactly once and are thus not again multiplied by S.

The unknowns in our system of equations are $H(N-2)S$ linear consumption functions (the functions for agents of age 1 do not appear in the equations) with $1 + H(N-2)$ coefficients each, S price functions for the Lucas-tree with $1 + H(N-2)$ coefficients each, S^2 Arrow security prices and $H(N-1)S^2$ portfolio variables for agents' holdings of Arrow securities for all possible combinations of s and s'.

To determine the coefficients of the linear policy and pricing functions we also need to vary the initial conditions. For each value of $\kappa^{a,h}$ we obtain another set of Equations (26)–(33). We do not increase the number of coefficients but only the number of Arrow security prices and portfolio variables. If we choose $1 + H(N - 2)$ affinely independent values for the initial conditions then we obtain a system with

$$(1 + H(N - 2)) \Big(S\left(H(N - 1)S + 1 + S \right) + H(N - 2)S \Big)$$
$$= (H(N - 1) + 1)(H(N - 2) + 1)S^2 + (H(N - 2) + 1)^2 S$$

equations and unknowns. A convenient choice for the initial conditions are the zero vector and all possible unit vectors for $\kappa^{a,h}$ for $a = 2, \ldots, N - 1$ and $h = 1, \ldots, H$. We denote this set of $1 + H(N - 2)$ values by $\mathcal{G}(\kappa)$.

For interesting model specifications the system of nonlinear equations becomes very large. For example, for $H = 2$, $N = 240$ and $S = 4$ the system consists of 4,565,844 equations and unknowns. Systems of such size are impossible to solve on a laptop without state-of-the-art software for Newton's method or some other algorithm for nonlinear equations. We solve these systems with a simple but slower iterative method based on a Jacobi scheme.

B.2 Iterative Jacobi method

At the beginning of an iteration, current iterates are available for the $S(1 + H(N - 2))$-dimensional coefficient vectors $\alpha^{a,h}$ and β. For each of the $S(1 + H(N - 2))$ possible combinations of $s \in \mathcal{S}$ and $\kappa \in \mathcal{G}(\kappa)$ we solve a linear system of equations. Observe that Equations (26)–(29) are linear in $q_{s'}$ and $q_{s'} \theta_{s'}^{a,h}$ for $s' \in \mathcal{S}$, $a = 1, \ldots, N - 1$, $h = 1, \ldots, H$. We can rewrite the market-clearing equations (31) as

$$\sum_{a=1}^{N-1} \sum_{h=1}^{H} q_{s'} \theta_{s'}^{a,h} = 0 \quad \text{for } s' = 1, \ldots, S. \tag{34}$$

The system (26)–(29) and (34) is a square linear system of $H(N - 1)S + S$ equations in the $H(N - 1)S$ unknowns $q_{s'} \theta_{s'}^{a,h}$ and the S unknowns $q_{s'}$. We solve this system with QR factorization with very small error (close to machine precision).

After we have solved $S(1 + H(N - 2))$ such systems of linear equations we can determine the new iterate for the coefficient vectors $\alpha^{a,h}$ and β. Note that after substituting all possible combinations of $s \in \mathcal{S}$ and $\kappa \in \mathcal{G}(\kappa)$ and the just computed accompanying solutions for $q_{s'}$ and $q_{s'} \theta_{s'}^{a,h}$ into Equations (30), (32), and (33) these in turn yield a system of $S(1 + H(N - 2))$ linear equations in the $S(1 + H(N - 2))$ unknown new coefficients $\alpha^{a,h}$ and β. The solution to this linear system replaces the current iterate for the coefficients and serves as the next iterate. Now a new iteration starts. This iterative procedure terminates when the infinity norm of two subsequent iterates falls below 10^{-10}.

B.3 Computation of Aggregate Statistics

In our model, the transition matrix Π determines the probability distribution over exogenous shocks in the next period. For a given shock in the following period the transition function of the endogenous state vector κ is deterministic. This mapping is not linear but the cash-at-hand of an agent in the next period can be written as the ratio of two linear functions of cash-at-hand across agents in the current period. This can be seen easily from the analysis above. Both the Arrow-prices q_s and the expenditure in Arrow-securities $q_s\theta_s$ are linear functions of the endogenous state κ. In our algorithm we therefore not only compute the pricing and consumption-policy coefficients, but we also compute the coefficients of the two functions that determine the transition. We can therefore easily simulate the economy and then compute the moments of interest.

We do not aim to numerically approximate the moments integrated over an invariant distribution. Since the transition function in our economy is not monotone, standard techniques for proving uniqueness of invariant distributions (see e.g. Bhattacharya and Majumdar (2007) for an overview) cannot be applied in our setting. Moreover, while we can do accuracy analysis for our Monte-Carlo approximations for a given finite horizon T, we do not know of any methods to do the same for the invariant distribution.

B.4 No closed-form solutions

The equilibrium equations (26)–(33) are a polynomial system. For small examples (e.g. $N = 3, 4$ and $H = 2$) we can therefore use Gröbner bases (see Kubler and Schmedders (2010)) to obtain an equivalent simpler system of equations and to solve for the unknown coefficients with very high precision. In addition, this approach enables us to explore under which conditions rational solutions to this system exist. For example, in the special case of a deterministic economy, if agents only have endowments in the first period of their lives, Huffman (1987) derives analytic and rational solutions for the coefficients. The Gröbner bases approach reveals that such simple expressions do not exist for more general models with agents having positive endowments for several periods. The coefficients are not rational expressions even if all exogenous parameters such as endowments, dividends, and the discount factor are rational numbers or integers.

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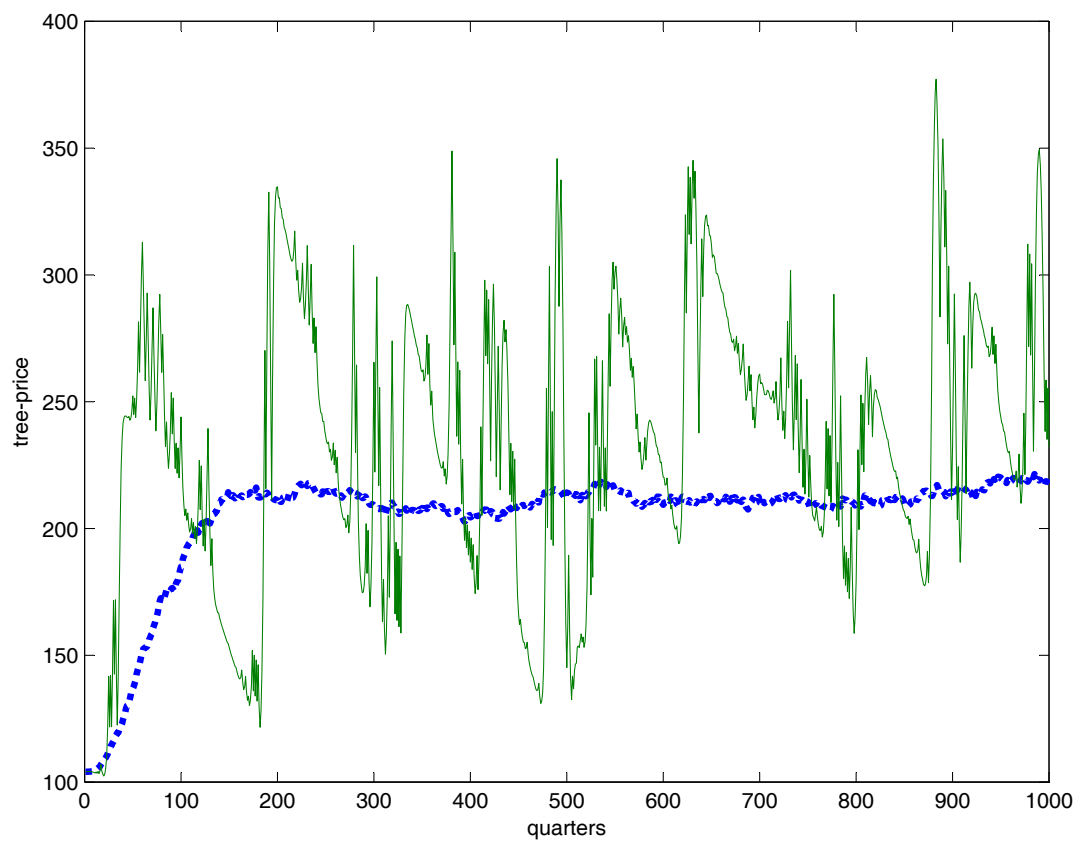
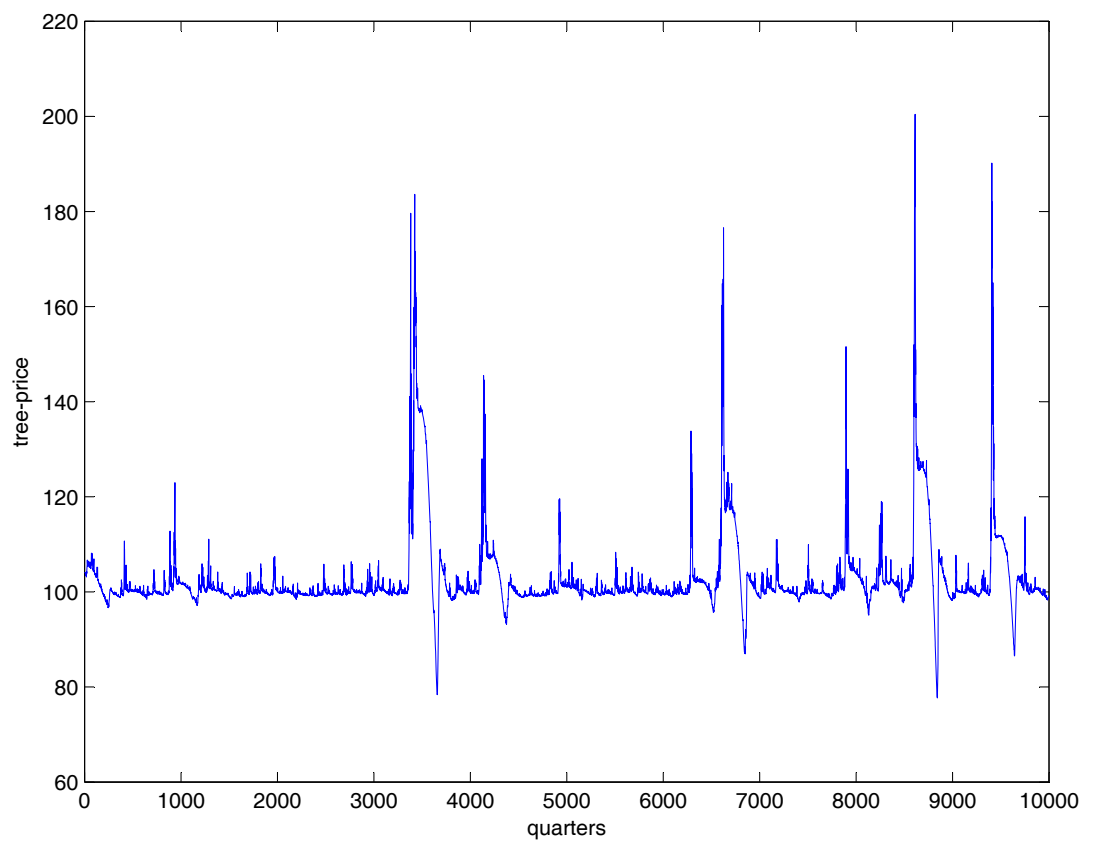Figure 1: Stock prices with persistently incorrect beliefs

Figure 2: Stock prices when most agents have correct beliefs

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