Numerical Optimization for Economists

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Part I

Numerical Optimization: Introduction
Modeling Language Benefits

- Portable language for optimization problems
  - Algebraic description
  - Models easily modified and solved
  - Large problems can be processed
  - Programming language features

- Many available optimization algorithms
- No need to compile C/FORTRAN code
- Derivatives automatically calculated
- Algorithms specific options can be set
- Communication with other tools
  - Relational databases and spreadsheets
  - MATLAB interface for function evaluations

- Excellent documentation
- Large user communities
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Modeling Languages Versus Solver Libraries

- Modeling languages
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  - Access to many numerical methods

- Solver libraries
  - Computationally efficient
  - Select appropriate matrix representation
  - Reorder data for locality of reference
  - Memory requirements are less
  - Very time consuming to code and validate application
  - Limited to a single numerical method
Modeling Languages Versus Solver Libraries

- **Modeling languages**
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Model Declaration

- **Sets**
  - Unordered, ordered, and circular sets
  - Cross products and point to set mappings
  - Set manipulation

- **Parameters and variables**
  - Initial and default values
  - Lower and upper bounds
  - Check statements
  - Defined variables

- **Objective function and constraints**
  - Equality, inequality, and range constraints
  - Complementarity constraints
  - Multiple objectives

- **Problem statement**
Data and Commands

- Data declaration
  - Set definitions
    - Explicit list of elements
    - Implicit list in parameter statements
  - Parameter definitions
    - Tables and transposed tables
    - Higher dimensional parameters
- Execution commands
  - Load model and data
  - Select problem, algorithm, and options
  - Solve the instance
  - Output results
- Other operations
  - Let and fix statements
  - Conditionals and loop constructs
  - Execution of external programs
Part II

Numerical Optimization I: Static Models
Model Formulation

- Classify $m$ people into two groups using $v$ variables
  - $c \in \{0, 1\}^m$ is the known classification
  - $d \in \mathbb{R}^{m \times v}$ are the observations
  - $\beta \in \mathbb{R}^{v+1}$ defines the separator
  - logit distribution function

- Maximum likelihood problem

$$\max_{\beta} \sum_{i=1}^{m} c_i \log(f(\beta, d_i,.)) + (1 - c_i) \log(1 - f(\beta, d_i,.))$$

where

$$f(\beta, x) = \frac{\exp \left( \beta_0 + \sum_{j=1}^{v} \beta_j x_j \right)}{1 + \exp \left( \beta_0 + \sum_{j=1}^{v} \beta_j x_j \right)}$$
Model: mle.mod

```plaintext
param m > 0, integer;    # Population
param v > 0, integer;    # Variables

param c {1..m} binary;  # Classification
param d {1..m, 1..v};   # Observation data

var beta {0..v};        # Weights

var lcomb {i in 1..m} = beta[0] + sum{j in 1..v} beta[j]*D[i,j];
var logit {i in 1..m} = exp(lcomb[i]) / (1+exp(lcomb[i]));

maximize likelihood:
    sum {i in 1..m} (c[i]*log(logit[i]) + (1-c[i])*log(1-logit[i]));
```

Munson Numerical Optimization
Data: mle.dat

param m := 3000; 
# Population
param v := 3;  
# Variables -- age, income, gender

# Random classification and observations
let {i in 1..m} c[i] := floor(Uniform(0,1) + 0.5);
let {i in 1..m, j in 1..v-1} d[i,j] := Uniform(0,1);
let {i in 1..m} d[i,v] := floor(Uniform(0,1) + 0.5);
# Load model and data
model mle.mod;
data mle.dat;

# Specify solver and options
option solver tron;
option tron_options "frtol=0 fatol=0 gtol=1e-8";

# Solve the instance
solve;

# Output the results
display beta;
ampl: include mle.cmd;
TRON: frtol=0
fatol=0
gtol=1e-8

Projected gradient at final iterate 5.95e-08
Function value at final iterate 2077.6373

Total execution time 0.03 sec

Exit Message CONVERGENCE: GTOL TEST SATISFIED

beta [*] :=
0  -0.186925
1   0.161719
2   0.15411
3   0.0509116
;
ampl: quit;
Solution Techniques

$$\min_x f(x)$$

Main ingredients of solution approaches:

- **Local method**: given $x_k$ (solution guess) compute a step $s$.
  - Gradient Descent
  - Quasi-Newton Approximation
  - Sequential Quadratic Programming
- **Globalization strategy**: converge from any starting point.
  - Trust region
  - Line search
Trust-Region Method

\[
\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T H(x_k) s \\
\text{subject to} \quad \|s\| \leq \Delta_k
\]
Trust-Region Method

1. Initialize trust-region radius
   - Constant
   - Direction
   - Interpolation
Trust-Region Method

1. Initialize trust-region radius
   - Constant
   - Direction
   - Interpolation

2. Compute a new iterate
   2.1 Solve trust-region subproblem
      \[
      \min_s \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T H(x_k) s \\
      \text{subject to} \quad \|s\| \leq \Delta_k
      \]
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2. Compute a new iterate
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      \[
      \min_s \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T H(x_k) s \\
      \text{subject to} \quad \|s\| \leq \Delta_k
      \]
   2.2 Accept or reject iterate
   2.3 Update trust-region radius
      - Reduction
      - Interpolation

3. Check convergence
Solving the Subproblem

- Moré-Sorensen method
  - Computes global solution to subproblem
- Conjugate gradient method with trust region
  - Objective function decreases monotonically
  - Some choices need to be made
    - Preconditioner
    - Norm of direction and residual
    - Dealing with negative curvature
Line-Search Method

\[
\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s
\]
Line-Search Method

1. Initialize perturbation to zero
2. Solve perturbed quadratic model

\[
\min_s f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s
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Line-Search Method

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\min_s \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2}s^T (H(x_k) + \lambda_k I)s
\]

3. Find new iterate
   3.1 Search along Newton direction
   3.2 Search along gradient-based direction

4. Update perturbation
   - Decrease perturbation if the following hold
     - Iterative method succeeds
     - Search along Newton direction succeeds
   - Otherwise increase perturbation
Line-Search Method

1. Initialize perturbation to zero
2. Solve perturbed quadratic model

\[
\min_s \ f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T (H(x_k) + \lambda_k I) s
\]

3. Find new iterate
   3.1 Search along Newton direction
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4. Update perturbation
   • Decrease perturbation if the following hold
     • Iterative method succeeds
     • Search along Newton direction succeeds
   • Otherwise increase perturbation
5. Check convergence
Solving the Subproblem

- Conjugate gradient method
Solving the Subproblem

- Conjugate gradient method
- Conjugate gradient method with trust region
  - Initialize radius
    - Constant
    - Direction
    - Interpolation
  - Update radius
    - Reduction
    - Step length
    - Interpolation
- Some choices need to be made
  - Preconditioner
  - Norm of direction and residual
  - Dealing with negative curvature
Performing the Line Search

- **Backtracking Armijo Line search**
  - Find \( t \) such that
    \[
    f(x_k + ts) \leq f(x_k) + \sigma t \nabla f(x_k)^T s
    \]
  - Try \( t = 1, \beta, \beta^2, \ldots \) for \( 0 < \beta < 1 \)

- **More-Thuente Line search**
  - Find \( t \) such that
    \[
    f(x_k + ts) \leq f(x_k) + \sigma t \nabla f(x_k)^T s
    \\
    |\nabla f(x_k + ts)^T s| \leq \delta |\nabla f(x_k)^T s|
    \]
  - Construct cubic interpolant
  - Compute \( t \) to minimize interpolant
  - Refine interpolant
Updating the Perturbation

1. If increasing and $\Delta^k = 0$

$$\Delta^{k+1} = \text{Proj}_{[\ell_0, u_0]} \left( \alpha_0 \| g(x^k) \| \right)$$

2. If increasing and $\Delta^k > 0$

$$\Delta^{k+1} = \text{Proj}_{[\ell_i, u_i]} \left( \max \left( \alpha_i \| g(x^k) \|, \beta_i \Delta^k \right) \right)$$

3. If decreasing

$$\Delta^{k+1} = \min \left( \alpha_d \| g(x^k) \|, \beta_d \Delta^k \right)$$

4. If $\Delta^{k+1} < \ell_d$, then $\Delta^{k+1} = 0$
Iterative Methods

• Conjugate gradient method
  • Stop if negative curvature encountered
  • Stop if residual norm is small
Iterative Methods

- Conjugate gradient method
  - Stop if negative curvature encountered
  - Stop if residual norm is small
- Conjugate gradient method with trust region
  - Nash
    - Follow direction to boundary if first iteration
    - Stop at base of direction otherwise
  - Steihaug-Toint
    - Follow direction to boundary
- Generalized Lanczos
  - Compute tridiagonal approximation
  - Find global solution to approximate problem on boundary
  - Initialize perturbation with approximate minimum eigenvalue
Preconditioners

- No preconditioner
- Absolute value of Hessian diagonal
- Absolute value of perturbed Hessian diagonal
- Incomplete Cholesky factorization of Hessian
- Block Jacobi with Cholesky factorization of blocks
- Scaled BFGS approximation to Hessian matrix
  - None
  - Scalar
  - Diagonal of Broyden update
  - Rescaled diagonal of Broyden update
  - Absolute value of Hessian diagonal
  - Absolute value of perturbed Hessian diagonal
Termination

- Typical convergence criteria
  - Absolute residual $\|\nabla f(x_k)\| < \tau_a$
  - Relative residual $\frac{\|\nabla f(x_k)\|}{\|\nabla f(x_k)\|} < \tau_r$
  - Unbounded objective $f(x_k) < \kappa$
  - Slow progress $|f(x_k) - f(x_{k-1})| < \epsilon$
  - Iteration limit
  - Time limit

- Solver status

```plaintext
display solve_result;  # String
display solve_result_num;  # Number
display $solve_result_table;  # Lookup table
```
Convergence Issues

- Quadratic convergence – best outcome
- Linear convergence
  - Far from a solution – $\|\nabla f(x_k)\|$ is large
  - Hessian is incorrect – disrupts quadratic convergence
  - Hessian is rank deficient – $\|\nabla f(x_k)\|$ is small
  - Limits of finite precision arithmetic
    1. $\|\nabla f(x_k)\|$ converges quadratically to small number
    2. $\|\nabla f(x_k)\|$ hovers around that number with no progress
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solution
  - Apply a multistart heuristic
  - Use global optimization solver
Some Available Software

- TRON – Newton method with trust-region
- LBFGS – Limited-memory quasi-Newton method with line search
- TAO – Toolkit for Advanced Optimization
  - NTR – Newton line-search method
  - NLS – Newton trust-region method
  - NTL – Newton line-search/trust-region method
  - LMVM – Limited-memory quasi-Newton method
  - CG – Nonlinear conjugate gradient methods
Model Formulation

- Economy with $n$ agents and $m$ commodities
  - $e \in \mathbb{R}^{n \times m}$ are the endowments
  - $\alpha \in \mathbb{R}^{n \times m}$ and $\beta \in \mathbb{R}^{n \times m}$ are the utility parameters
  - $\lambda \in \mathbb{R}^{n}$ are the social weights

- Social planning problem

$$\max_{x \geq 0} \sum_{i=1}^{n} \lambda_i \left( \sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}} \right)$$

subject to

$$\sum_{i=1}^{n} x_{i,k} \leq \sum_{i=1}^{n} e_{i,k} \quad \forall k = 1, \ldots, m$$
Model: social1.mod

param n > 0, integer; # Agents
param m > 0, integer; # Commodities

param e {1..n, 1..m} >= 0, default 1; # Endowment

param lambda {1..n} > 0; # Social weights
param alpha {1..n, 1..m} > 0; # Utility parameters
param beta {1..n, 1..m} > 0;
Model: social1.mod

param n > 0, integer;  # Agents
param m > 0, integer;  # Commodities

param e {1..n, 1..m} >= 0, default 1;  # Endowment

param lambda {1..n} > 0;  # Social weights
param alpha {1..n, 1..m} > 0;  # Utility parameters
param beta {1..n, 1..m} > 0;

var x{1..n, 1..m} >= 0;  # Consumption
var u{i in 1..n} =
    sum {k in 1..m} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);
Model: social1.mod

param n > 0, integer;                        # Agents
param m > 0, integer;                        # Commodities

param e {1..n, 1..m} >= 0, default 1;       # Endowment
param lambda {1..n} > 0;                    # Social weights
param alpha {1..n, 1..m} > 0;               # Utility parameters
param beta {1..n, 1..m} > 0;

var x{1..n, 1..m} >= 0;                    # Consumption
var u{i in 1..n} =
    sum {k in 1..m} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);

maximize welfare:
    sum {i in 1..n} lambda[i] * u[i];

subject to
    consumption {k in 1..m}:
        sum {i in 1..n} x[i,k] <= sum {i in 1..n} e[i,k];
Data: social1.dat

param n := 3;  # Agents
param m := 4;  # Commodities
### Data: social1.dat

param n := 3;  # Agents
param m := 4;  # Commodities

**param alpha:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>5</td>
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</table>

**param beta (tr):**

<table>
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<tbody>
<tr>
<td>1</td>
<td>1.5</td>
<td>2</td>
<td>0.6</td>
</tr>
<tr>
<td>2</td>
<td>1.6</td>
<td>3</td>
<td>0.7</td>
</tr>
<tr>
<td>3</td>
<td>1.7</td>
<td>2</td>
<td>2.0</td>
</tr>
<tr>
<td>4</td>
<td>1.8</td>
<td>2</td>
<td>2.5</td>
</tr>
</tbody>
</table>

**param : lambda:**

<p>| | |</p>
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<tbody>
<tr>
<td>1</td>
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</tr>
<tr>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>
# Load model and data
model social1.mod;
data social1.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in 1..n} "%2d: % 5.4e\n", i, u[i];
Output

ampl: include social1.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.25422003
Nonlin evals: obj = 44, grad = 43.
x :=
  1 1  0.0811471
  1 2  0.574164
  1 3  0.703454
  1 4  0.267241
  2 1  0.060263
  2 2  0.604858
  2 3  1.7239
  2 4  1.47516
  3 1  2.85859
  3 2  1.82098
  3 3  0.572645
  3 4  1.2576
;

  1:  -5.2111e+00
  2:  -4.0488e+00
  3:   1.1512e+01
ampl: quit;
Model: social2.mod

set AGENTS; # Agents
set COMMODITIES; # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment
param lambda {AGENTS} > 0; # Social weights
param alpha {AGENTS, COMMODITIES} > 0; # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;
param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];
var x{AGENTS, COMMODITIES} >= 0; # Consumption
var u{i in AGENTS} = # Utility
    sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^gamma[i,k] / gamma[i,k];
maximize welfare:
    sum {i in AGENTS} lambda[i] * u[i];
subject to
    consumption {k in COMMODITIES}:
        sum {i in AGENTS} x[i,k] <= sum {i in AGENTS} e[i,k];
Model: social2.mod

set AGENTS;                    # Agents
set COMMODITIES;               # Commodities

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param lambda {AGENTS} > 0;     # Social weights
param alpha {AGENTS, COMMODITIES} > 0; # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];

var x {AGENTS, COMMODITIES} >= 0; # Consumption

var u {i in AGENTS} = # Utility
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Model: social2.mod

set AGENTS;    # Agents
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param lambda {AGENTS} > 0;   # Social weights
param alpha {AGENTS, COMMODITIES} > 0; # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

param gamma {i in AGENTS, k in COMMODITIES} := 1 - beta[i,k];

var x{AGENTS, COMMODITIES} >= 0; # Consumption
var u{i in AGENTS} = # Utility
    sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^gamma[i,k] / gamma[i,k];

maximize welfare:
    sum {i in AGENTS} lambda[i] * u[i];

subject to
    consumption {k in COMMODITIES}:
        sum {i in AGENTS} x[i,k] <= sum {i in AGENTS} e[i,k];
Data: social2.dat

set COMMODITIES := Books, Cars, Food, Pens;

param: AGENTS : lambda :=
    Jorge 1
    Sven  1
    Todd  1;

param alpha : Books Cars Food Pens :=
    Jorge 1 1 1 1
    Sven  1 2 3 4
    Todd  2 1 1 5;

param beta (tr): Jorge Sven Todd :=
    Books 1.5 2 0.6
    Cars 1.6 3 0.7
    Food 1.7 2 2.0
    Pens 1.8 2 2.5;
Data: social2.dat

set COMMODITIES := Books, Cars, Food, Pens;

param: AGENTS : lambda :=
    Jorge    1
    Sven     1
    Todd     1;

param alpha : Books   Cars   Food   Pens :=
    Jorge    1   1   1   1
    Sven     1   2   3   4
    Todd     2   1   1   5;

param beta (tr): Jorge  Sven  Todd :=
    Books    1.5  2   0.6
    Cars     1.6  3   0.7
    Food     1.7  2   2.0
    Pens     1.8  2   2.5;
Commands: social2.cmd

# Load model and data
model social2.mod;
data social2.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
display x;
printf {i in AGENTS} "%-5s: %g
", i, u[i];
Output

ampl: include social2.cmd
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
25 iterations, objective 2.25422003
Nonlin evals: obj = 44, grad = 43.
x :=
    Jorge Books  0.0811471
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    Sven Pens     1.47516
    Todd Books    2.85859
    Todd Cars     1.82098
    Todd Food     0.572645
    Todd Pens     1.2576
;
Jorge: -5.2111e+00
   Sven: -4.0488e+00
   Todd:  1.1512e+01
ampl: quit;
Solving Constrained Optimization Problems

\[ \min_{x} f(x) \]

subject to \( c(x) \geq 0 \)

Main ingredients of solution approaches:

- Local method: given \( x_k \) (solution guess) find a step \( s \).
  - Sequential Quadratic Programming (SQP)
  - Sequential Linear/Quadratic Programming (SLQP)
  - Interior-Point Method (IPM)

- Globalization strategy: converge from any starting point.
  - Trust region
  - Line search

- Acceptance criteria: filter or penalty function.
Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate
Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program

\[
\begin{align*}
\min_{s} & \quad f(x_k) + s^T \nabla f(x_k) \\
\text{subject to} & \quad c(x_k) + \nabla c(x_k)^T s \geq 0 \\
& \quad \|s\| \leq \Delta_k
\end{align*}
\]
Sequential Linear Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program
   \[
   \begin{align*}
   \min_s & \quad f(x_k) + s^T \nabla f(x_k) \\
   \text{subject to} & \quad c(x_k) + \nabla c(x_k)^T s \geq 0 \\
   & \quad \|s\| \leq \Delta_k
   \end{align*}
   \]
   2.2 Accept or reject iterate
   2.3 Update trust-region radius
3. Check convergence
Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve quadratic program

\[
\min_{s} \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\
\text{subject to} \quad c(x_k) + \nabla c(x_k)^T s \geq 0 \\
\|s\| \leq \Delta_k
\]
Sequential Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve quadratic program
   \[
   \min_s \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\
   \text{subject to} \quad c(x_k) + \nabla c(x_k)^T s \geq 0 \\
   \|s\| \leq \Delta_k
   \]
   2.2 Accept or reject iterate
   2.3 Update trust-region radius
3. Check convergence
Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program to predict active set

\[
\begin{align*}
\min_{\mathbf{d}} & \quad f(x_k) + \mathbf{d}^T \nabla f(x_k) \\
\text{subject to} & \quad c(x_k) + \nabla c(x_k)^T \mathbf{d} \geq 0 \\
& \quad \|\mathbf{d}\| \leq \Delta_k
\end{align*}
\]
Sequential Linear Quadratic Programming

1. Initialize trust-region radius
2. Compute a new iterate
   2.1 Solve linear program to predict active set
      \[
      \min_d \quad f(x_k) + d^T \nabla f(x_k) \\
      \text{subject to } c(x_k) + \nabla c(x_k)^T d \geq 0 \\
      \|d\| \leq \Delta_k
      \]
   2.2 Solve equality constrained quadratic program
      \[
      \min_s \quad f(x_k) + s^T \nabla f(x_k) + \frac{1}{2} s^T W(x_k) s \\
      \text{subject to } c_A(x_k) + \nabla c_A(x_k)^T s = 0
      \]
   2.3 Accept or reject iterate
   2.4 Update trust-region radius
3. Check convergence
Acceptance Criteria

- Decrease objective function value: \( f(x_k + s) \leq f(x_k) \)
- Decrease constraint violation: \( \|c_-(x_k + s)\| \leq \|c_-(x_k)\| \)
Acceptance Criteria

- Decrease objective function value: \( f(x_k + s) \leq f(x_k) \)
- Decrease constraint violation: \( \| c_-(x_k + s) \| \leq \| c_-(x_k) \| \)
- Four possibilities
  1. step can decrease both \( f(x) \) and \( \| c_-(x) \| \) GOOD
  2. step can decrease \( f(x) \) and increase \( \| c_-(x) \| \) ???
  3. step can increase \( f(x) \) and decrease \( \| c_-(x) \| \) ???
  4. step can increase both \( f(x) \) and \( \| c_-(x) \| \) BAD

Filter uses concept from multi-objective optimization
\((h_k + 1, f_{k+1}) \) dominates \((h_\ell, f_\ell)\) iff \( h_k + 1 \leq h_\ell \) and \( f_{k+1} \leq f_\ell \)
Acceptance Criteria

• Decrease objective function value: \( f(x_k + s) \leq f(x_k) \)
• Decrease constraint violation: \( \|c_-(x_k + s)\| \leq \|c_-(x_k)\| \)
• Four possibilities
  1. step can decrease both \( f(x) \) and \( \|c_-(x)\| \) GOOD
  2. step can decrease \( f(x) \) and increase \( \|c_-(x)\| \) ???
  3. step can increase \( f(x) \) and decrease \( \|c_-(x)\| \) ???
  4. step can increase both \( f(x) \) and \( \|c_-(x)\| \) BAD

• Filter uses concept from multi-objective optimization

\((h_{k+1}, f_{k+1})\) dominates \((h_{\ell}, f_{\ell})\) iff \( h_{k+1} \leq h_{\ell} \) and \( f_{k+1} \leq f_{\ell} \)
Filter Framework

Filter $\mathcal{F}$: list of non-dominated pairs $(h_\ell, f_\ell)$

- new $x_{k+1}$ is acceptable to filter $\mathcal{F}$ iff
  1. $h_{k+1} \leq h_\ell$ for all $\ell \in \mathcal{F}$ or
  2. $f_{k+1} \leq f_\ell$ for all $\ell \in \mathcal{F}$
Filter Framework

Filter $\mathcal{F}$: list of non-dominated pairs $(h_\ell, f_\ell)$

- new $x_{k+1}$ is acceptable to filter $\mathcal{F}$ iff
  1. $h_{k+1} \leq h_\ell$ for all $\ell \in \mathcal{F}$ or
  2. $f_{k+1} \leq f_\ell$ for all $\ell \in \mathcal{F}$
- remove redundant filter entries
Filter Framework

Filter $\mathcal{F}$: list of non-dominated pairs $(h_\ell, f_\ell)$

- new $x_{k+1}$ is acceptable to filter $\mathcal{F}$ iff
  1. $h_{k+1} \leq h_\ell$ for all $\ell \in \mathcal{F}$ or
  2. $f_{k+1} \leq f_\ell$ for all $\ell \in \mathcal{F}$

- remove redundant filter entries

- new $x_{k+1}$ is rejected if for some $\ell \in \mathcal{F}$
  1. $h_{k+1} > h_\ell$ and
  2. $f_{k+1} > f_\ell$
Convergence Criteria

- Feasible and no descent directions
  - Constraint qualification – LICQ, MFCQ
  - Linearized active constraints characterize directions
  - Objective gradient is a linear combination of constraint gradients
Optimality Conditions

- If $x^*$ is a local minimizer and a constraint qualification holds, then there exist multipliers $\lambda^* \geq 0$ such that
  \[ \nabla f(x^*) - \nabla c_A(x^*)^T \lambda^*_A = 0 \]

- Lagrangian function $L(x, \lambda) := f(x) - \lambda^T c(x)$

- Optimality conditions can be written as
  \[ \nabla f(x) - \nabla c(x)^T \lambda = 0 \]
  \[ 0 \leq \lambda \perp c(x) \geq 0 \]

- Complementarity problem
Termination

- Feasible and complementary \( \| \text{min}(c(x_k), \lambda_k) \| \leq \tau_f \)
- Optimal \( \| \nabla_x \mathcal{L}(x_k, \lambda_k) \| \leq \tau_o \)
- Other possible conditions
  - Slow progress
  - Iteration limit
  - Time limit
- Multipliers and reduced costs

```
display consumption.slack; # Constraint violation
display consumption.dual;   # Lagrange multipliers
display x.rc;               # Gradient of Lagrangian
```
Convergence Issues

- Quadratic convergence – best outcome
- Globally infeasible – linear constraints infeasible
- Locally infeasible – nonlinear constraints locally infeasible
- Unbounded objective – hard to detect
- Unbounded multipliers – constraint qualification not satisfied
- Linear convergence rate
  - Far from a solution – $\|\nabla f(x_k)\|$ is large
  - Hessian is incorrect – disrupts quadratic convergence
  - Hessian is rank deficient – $\|\nabla f(x_k)\|$ is small
  - Limits of finite precision arithmetic
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solutions
  - Apply a multistart heuristic
  - Use global optimization solver
Some Available Software

- ASTROS – Active-Set Trust-Region Optimization Solvers
- filterSQP
  - trust-region SQP; robust QP solver
  - filter to promote global convergence
- SNOPT
  - line-search SQP; null-space CG option
  - $\ell_1$ exact penalty function
- SLIQUE – part of KNITRO
  - SLP-EQP
  - trust-region with $\ell_1$ penalty
  - use with `knitro_options = "algorithm=3";`
Model Formulation

- Maximize discounted utility
  - $u(\cdot)$ is the utility function
  - $R$ is the retirement age
  - $T$ is the terminal age
  - $w$ is the wage
  - $\beta$ is the discount factor
  - $r$ is the interest rate

- Optimization problem

$$\max_{s,c} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$s_{t+1} = (1 + r)s_t + w - c_t \quad t = 0, \ldots, R - 1$$
$$s_{t+1} = (1 + r)s_t - c_t \quad t = R, \ldots, T$$
$$s_0 = s_{T+1} = 0$$
Model: life1.mod

param R > 0, integer; # Retirement age
param T > R, integer; # Terminal age
param beta >= 0, < 1; # Discount factor
param rate >= 0, < 1; # Interest rate
param wage >= 0; # Wage rate

var c{0..T}; # Consumption
var s{0..T+1}; # Savings
var u{t in 0..T} = -exp(-c[t]); # Utility

maximize utility:
sum {t in 0..T} beta^t * u[t];

subject to
working {t in 0..R-1}:
s[t+1] = (1+rate)*s[t] + wage - c[t];
retired {t in R..T}:
s[t+1] = (1+rate)*s[t] - c[t];
initial:
s[0] = 0;
terminal:
s[T+1] = 0;
Model: life1.mod

\begin{verbatim}
param R > 0, integer;  # Retirement age
param T > R, integer;  # Terminal age

param beta >= 0, < 1;  # Discount factor
param rate >= 0, < 1;  # Interest rate
param wage >= 0;      # Wage rate

var c{0..T};          # Consumption
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  \sum {t in 0..T} beta^t * u[t];
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  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - c[t];
  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - c[t];
  initial:
    s[0] = 0;
  terminal:
    s[T+1] = 0;
\end{verbatim}
Model: life1.mod

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maximize utility:
  sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - c[t];

  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - c[t];

  initial:
    s[0] = 0;

  terminal:
    s[T+1] = 0;
Data: life1.dat

param R := 7; # Retirement age
param T := 10; # Terminal age
param beta := 0.9; # Discount factor
param rate := 0.2; # Interest rate
param wage := 1.0; # Wage rate
Commands: life1.cmd

# Load model and data
model life1.mod;
data life1.dat;

# Specify solver and options
option solver mpec;

# Solve the instance
solve;

# Output results
printf {t in 0..T} "%2d %5.4e %5.4e\n", t, s[t], c[t] > out1.dat;
Output

ampl: include life1.cmd
AMPL interface to filter-MPEC: 20040408
  : filter objective function  = -3.24322
       constraint violation = 1.01433e-11
Optimal solution found
14 iterations (0 for feasibility)
Evals: obj = 15, constr = 16, grad = 16, Hes = 15
ampl: quit;
Model: negishi.mod

set Commodities; # Goods and factors
set Producers within Commodities; # Consumer goods produced
Model: negishi.mod

set Commodities; # Goods and factors
set Producers within Commodities; # Consumer goods produced

##### Structure of the Nested CES Functions for Producers ######
# INodes: internal nodes for the tree in postfix order
# FUse : factor inputs used by the node
# IUse : internal node output used by the node
# TUse : full list of factors referenced

set P_INodes {Producers} ordered;
set P_FUse {p in Producers, P_INodes[p]} within Commodities;
set P_IUse {p in Producers, P_INodes[p]} within P_INodes[p];
set P_TUse {p in Producers} := union {k in P_INodes[p]} P_FUse[p,k];
Model: negishi.mod

set Commodities;          # Goods and factors
set Producers within Commodities;  # Consumer goods produced

##### Structure of the Nested CES Functions for Producers ######
# INodes: internal nodes for the tree in postfix order
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set P_FUse {p in Producers, P_INodes[p]} within Commodities;
set P_IUse {p in Producers, P_INodes[p]} within P_INodes[p];
set P_TUse {p in Producers} := union {k in P_INodes[p]} P_FUse[p,k];

##### Parameters for Producer Nested CES Functions #######

param P_Scale {p in Producers, i in P_INodes[p]}
param P_Elasticity {p in Producers, i in P_INodes[p]}
param P_FDist {p in Producers, i in P_INodes[p], k in P_FUse[p,i]}
param P_IDist {p in Producers, i in P_INodes[p], j in P_IUse[p,i]}

param P_Alpha {p in Producers, i in P_INodes[p]}
  := 1.0 - 1.0/P_Elasticity[p,i];
param P_Beta {p in Producers, i in P_INodes[p]} := 1.0 / P_Alpha[p,i];
### Producer Behavior

# Input : quantities of commodities demanded by the producers
# Output: quantities of commodities created by the producers

```plaintext
var Input {p in Producers, P_TUse[p]} >= 0, := 1;

var Output {p in Producers, i in P_INodes[p]} =
    P_Scale[p,i]*(
        sum {k in P_FUse[p,i]} P_FDist[p,i,k]*Input[p,k]^P_Alpha[p,i] +
        sum {j in P_IUse[p,i]} P_IDist[p,i,j]*Output[p,j]^P_Alpha[p,i]
    )^P_Beta[p,i];
```

### Consumer Behavior

# Endow : initial endowment of commodities
# Demand : quantities of commodities demanded by the consumer
# Utility: utility of the commodities demanded by the consumer
# : determined from the nested CES Utility Function

```plaintext
param Endow {c in Consumers, k in Commodities};
var Demand {c in Consumers, C_TUse[c]} >= 0, := 1;

var Utility {c in Consumers, i in C_INodes[c]} =
    C_Scale[c,i]*(
        sum {k in C_FUse[c,i]} C_AFDist[c,i,k]*Demand[c,k]^C_Alpha[c,i] +
        sum {j in C_IUse[c,i]} C_AIDist[c,i,j]*Utility[c,j]^C_Alpha[c,i]
    )^C_Beta[c,i];
```
Model: negishi.mod

##### Producer Behavior ######
# Input: quantities of commodities demanded by the producers
# Output: quantities of commodities created by the producers

var Input \{p \in Producers, P\_TUse[p]\} \geq 0, := 1;

var Output \{p \in Producers, i \in P\_INodes[p]\} = 
P\_Scale[p,i]*(
  \sum \{k \in P\_FUse[p,i]\} P\_FDist[p,i,k]*Input[p,k]^P\_Alpha[p,i] +
  \sum \{j \in P\_IUse[p,i]\} P\_IDist[p,i,j]*Output[p,j]^P\_Alpha[p,i]
)^{P\_Beta[p,i]};

##### Consumer Behavior ######
# Endow: initial endowment of commodities
# Demand: quantities of commodities demanded by the consumer
# Utility: utility of the commodities demanded by the consumer
# : determined from the nested CES Utility Function

param Endow \{c \in Consumers, k \in Commodities\};

var Demand \{c \in Consumers, C\_TUse[c]\} \geq 0, := 1;

var Utility \{c \in Consumers, i \in C\_INodes[c]\} = 
C\_Scale[c,i]*(
  \sum \{k \in C\_FUse[c,i]\} C\_AFDist[c,i,k]*Demand[c,k]^C\_Alpha[c,i] +
  \sum \{j \in C\_IUse[c,i]\} C\_AIDist[c,i,j]*Utility[c,j]^C\_Alpha[c,i]
)^{C\_Beta[c,i]};
Model: negishi.mod

###### Negishi Weights ######

```
param Weights{c in Consumers} default 1 / card(Consumers);
```

###### Negishi Optimization Problem ######

```
maximize Welfare:
  sum{c in Consumers} Weights[c]*Utility[c,last(C_INodes[c])];

subject to

Market {k in Commodities}:
  sum{c in Consumers} Endow[c,k] + (if (k in Producers) then Output[k,last(P_INodes[k])]) >=
  sum{p in Producers: k in P_TUse[p]} Input[p,k] + sum{c in Consumers: k in C_TUse[c]} Demand[c,k];
```
set Commodities := G1 G2 F1 F2 F3;
set Consumers := C1 C2;
set Producers := G1 G2;
set Numeraire := F1;

Data: negishi.dat

set Commodities := G1 G2 F1 F2 F3;
set Consumers := C1 C2;
set Producers := G1 G2;
set Numeraire := F1;

set P_INodes[G1] := I1 I2 I3;
set P_FUse[G1, I1] := F1 F2;
set P_FUse[G1, I2] := F1 F3;
set P_FUse[G1, I3] := F2 F3 G2;
set P_IUse[G1, I1] := ;
set P_IUse[G1, I2] := ;
set P_IUse[G1, I3] := I1 I2;
set P_INodes[G2] := I1 I2;
set P_FUse[G2, I1] := F1 F2;
set P_FUse[G2, I2] := F3 G1;
set P_IUse[G2, I1] := ;
set P_IUse[G2, I2] := I1;
Model Formulation

- Route commodities through a network
  - \( \mathcal{N} \) is the set of nodes
  - \( \mathcal{A} \subseteq \mathcal{N} \times \mathcal{N} \) is the set of arcs
  - \( \mathcal{K} \) is the set of commodities
  - \( \alpha \) and \( \beta \) are the congestion parameters
  - \( b \) denotes the supply and demand

- Multicommodity network flow problem

\[
\begin{align*}
\text{max} & \quad \sum_{(i,j) \in \mathcal{A}} \left( \alpha_{i,j} f_{i,j} + \beta_{i,j} f_{i,j}^4 \right) \\
\text{subject to} & \quad \sum_{(i,j) \in \mathcal{A}} x_{i,j,k} \leq \sum_{(j,i) \in \mathcal{A}} x_{j,i,k} + b_{i,k} \quad \forall i \in \mathcal{N}, k \in \mathcal{K} \\
& \quad f_{i,j} = \sum_{k \in \mathcal{K}} x_{i,j,k} \quad \forall (i, j) \in \mathcal{A}
\end{align*}
\]
Model: network.mod

set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES := 1..3;

# Nodes in network
# Arics in network
# Commodities
Model: network.mod

set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES := 1..3;

param b {NODES, COMMODITIES} default 0; # Supply/demand
check {k in COMMODITIES}:
  sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0; # Linear part
param beta{ARCS} >= 0; # Nonlinear part

var x{ARCS, COMMODITIES} >= 0; # Flow on arcs
var f{(i,j) in ARCS} = # Total flow
  sum {k in COMMODITIES} x[i,j,k];

minimize time:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];
Model: network.mod

```plaintext
set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES := 1..3;

param b {NODES, COMMODITIES} default 0;
check {k in COMMODITIES}:
  sum {i in NODES} b[i,k] >= 0;

param alpha {ARCS} >= 0;
param beta {ARCS} >= 0;

var x {ARCS, COMMODITIES} >= 0;
var f {(i,j) in ARCS} =
  sum {k in COMMODITIES} x[i,j,k];

minimize time:
  sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);
subject to
  conserve {i in NODES, k in COMMODITIES}:
    sum {(i,j) in ARCS} x[i,j,k] <= sum {(j,i) in ARCS} x[j,i,k] + b[i,k];
```

# Nodes in network
# Arcs in network
# Commodities

# Supply/demand
# Supply exceeds demand

# Linear part
# Nonlinear part

# Flow on arcs
# Total flow
Model: network.mod

set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES := 1..3;

param b {NODES, COMMODITIES} default 0;
check {k in COMMODITIES}:
    sum{i in NODES} b[i,k] >= 0;

param alpha{ARCS} >= 0;
param beta{ARCS} >= 0;

var x{ARCS, COMMODITIES} >= 0;
var f{(i,j) in ARCS} =
    sum {k in COMMODITIES} x[i,j,k];

minimize time:
    sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
    conserve {i in NODES, k in COMMODITIES}:
        sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];
Data: network.dat

set NODES := 1 2 3 4 5;

param: ARCS : alpha beta =
          1 2  1  0.5
          1 3  1  0.4
          2 3  2  0.7
          2 4  3  0.1
          3 2  1  0.0
          3 4  4  0.5
          4 1  5  0.0
          4 5  2  0.1
          5 2  0  1.0;
Data: network.dat

set NODES := 1 2 3 4 5;

param: ARCS : alpha beta =
    1 2  1  0.5
    1 3  1  0.4
    2 3  2  0.7
    2 4  3  0.1
    3 2  1  0.0
    3 4  4  0.5
    4 1  5  0.0
    4 5  2  0.1
    5 2  0  1.0;

let b[1,1] := 7;    # Node 1, Commodity 1 supply
let b[4,1] := -7;  # Node 4, Commodity 1 demand
let b[2,2] := 3;   # Node 2, Commodity 2 supply
let b[5,2] := -3;  # Node 5, Commodity 2 demand
let b[3,3] := 5;   # Node 1, Commodity 3 supply
let b[1,3] := -5;  # Node 4, Commodity 3 demand

fix {i in NODES, k in COMMODITIES: (i,i) in ARCS} x[i,i,k] := 0;
Commands: network.cmd

# Load model and data
model network.mod;
data network.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Solve the instance
solve;

# Output results
for {k in COMMODITIES} {
    printf "Commodity: %d\n", k > network.out;
    printf \{(i,j) in ARCS: x[i,j,k] > 0\} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > network.out;
    printf "\n" > network.out;
}
Output

ampl: include network.cmd;
MINOS 5.5: outlev=1
MINOS 5.5: optimal solution found.
12 iterations, objective 1505.526478
Nonlin evals: obj = 14, grad = 13.
ampl: quit;
Results: network.out

Commodity: 1
1.2 = 3.3775e+00
1.3 = 3.6225e+00
2.4 = 6.4649e+00
3.2 = 3.0874e+00
3.4 = 5.3510e-01

Commodity: 2
2.4 = 3.0000e+00
4.5 = 3.0000e+00

Commodity: 3
3.4 = 5.0000e+00
4.1 = 5.0000e+00
Initial Coordinate Descent: wardrop0.cmd

# Load model and data
model network.mod;
data network.dat;

option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
fix {(i,j) in ARCS, k in COMMODITIES} x[i,j,k];
drop {i in NODES, k in COMMODITIES} conserve[i,k];

for {iter in 1..100} {
    for {k in COMMODITIES} {
        unfix {(i,j) in ARCS} x[i,j,k];
        restore {i in NODES} conserve[i,k];

        solve;

        fix {(i,j) in ARCS} x[i,j,k];
        drop {i in NODES} conserve[i,k];
    }
}

# Output results
for {k in COMMODITIES} {
    printf "\nCommodity: %d\n", k > network.out;
    printf {(i,j) in ARCS: x[i,j,k] > 0} "\%d.\%d = \% 5.4e\n", i, j, x[i,j,k] > network.out;
}
Improved Coordinate Descent: wardrop.mod

```plaintext
set NODES;
set ARCS within NODES cross NODES;
set COMMODITIES := 1..3;

param b {NODES, COMMODITIES} default 0; # Supply/demand
param alpha {ARCS} >= 0; # Linear part
param beta {ARCS} >= 0; # Nonlinear part

var x {ARCS, COMMODITIES} >= 0; # Flow on arcs
var f {(i,j) in ARCS} = sum {k in COMMODITIES} x[i,j,k]; # Total flow

minimize time {k in COMMODITIES}:
    sum {(i,j) in ARCS} (alpha[i,j]*f[i,j] + beta[i,j]*f[i,j]^4);

subject to
    conserve {i in NODES, k in COMMODITIES}:
        sum {(i,j) in ARCS} x[i,j,k] <= sum{(j,i) in ARCS} x[j,i,k] + b[i,k];

problem subprob {k in COMMODITIES}: time[k], {i in NODES} conserve[i,k],
    {(i,j) in ARCS} x[i,j,k], f;
```
Improved Coordinate Descent: wardrop1.cmd

```plaintext
# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
for {iter in 1..100} {
    for {k in COMMODITIES} {
        solve subprob[k];
    }
}

for {k in COMMODITIES} {
    printf "Commodity: %d\n", k > wardrop.out;
    printf \{(i,j) in ARCS: x[i,j,k] > 0\} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
    printf "\n" > wardrop.out;
}
```
Final Coordinate Descent: wardrop2.cmd

# Load model and data
model wardrop.mod;
data wardrop.dat;

# Specify solver and options
option solver minos;
option minos_options "outlev=1";

# Coordinate descent method
param xold{ARCS, COMMODITIES};
param xnew{ARCS, COMMODITIES};

repeat {
  for {k in COMMODITIES} {
    problem subprob[k];
    let {(i,j) in ARCS} xold[i,j,k] := x[i,j,k];
    solve;
    let {(i,j) in ARCS} xnew[i,j,k] := x[i,j,k];
  }
} until (sum {(i,j) in ARCS, k in COMMODITIES} abs(xold[i,j,k] - xnew[i,j,k]) <= 1e-6);

for {k in COMMODITIES} {
  printf "Commodity: %d\n", k > wardrop.out;
  printf {(i,j) in ARCS: x[i,j,k] > 0} "%d.%d = % 5.4e\n", i, j, x[i,j,k] > wardrop.out;
  printf "\n" > wardrop.out;
}
Part III

Numerical Optimization II: Optimal Control
Model Formulation

- Maximize discounted utility
  - $u(\cdot)$ is the utility function
  - $R$ is the retirement age
  - $T$ is the terminal age
  - $w$ is the wage
  - $\beta$ is the discount factor
  - $r$ is the interest rate

- Optimization problem

\[
\max_{s,c} \sum_{t=0}^{T} \beta^t u(c_t)
\]

subject to

\[
\begin{align*}
  s_{t+1} &= (1 + r)s_t + w - c_t & t &= 0, \ldots, R - 1 \\
  s_{t+1} &= (1 + r)s_t - c_t & t &= R, \ldots, T \\
  s_0 &= s_{T+1} = 0
\end{align*}
\]
Model: life1.mod

param R > 0, integer; # Retirement age
param T > R, integer; # Terminal age
param beta >= 0, < 1; # Discount factor
param rate >= 0, < 1; # Interest rate
param wage >= 0; # Wage rate

var c{0..T}; # Consumption
var s{0..T+1}; # Savings
var u{t in 0..T} = -exp(-c[t]); # Utility

maximize utility:
  sum {t in 0..T} beta^t * u[t];

subject to
  working {t in 0..R-1}:
    s[t+1] = (1+rate)*s[t] + wage - c[t];
  retired {t in R..T}:
    s[t+1] = (1+rate)*s[t] - c[t];
  initial:
    s[0] = 0;
  terminal:
    s[T+1] = 0;
Model: life1.mod

param R > 0, integer;                     # Retirement age
param T > R, integer;                    # Terminal age

param beta >= 0, < 1;                    # Discount factor
param rate >= 0, < 1;                    # Interest rate
param wage >= 0;                         # Wage rate

var c{0..T};                             # Consumption
var s{0..T+1};                           # Savings
var u{t in 0..T} = -exp(-c[t]);          # Utility

maximize utility:
    sum {t in 0..T} beta^t * u[t];
subject to
    working {t in 0..R-1}:
        s[t+1] = (1+rate)*s[t] + wage - c[t];
    retired {t in R..T}:
        s[t+1] = (1+rate)*s[t] - c[t];
    initial:
        s[0] = 0;
    terminal:
        s[T+1] = 0;
Model: life1.mod

param R > 0, integer; # Retirement age
param T > R, integer; # Terminal age
param beta >= 0, < 1; # Discount factor
param rate >= 0, < 1; # Interest rate
param wage >= 0; # Wage rate

var c{0..T}; # Consumption
var s{0..T+1}; # Savings
var u{t in 0..T} = -exp(-c[t]); # Utility

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
    working {t in 0..R-1}:
        s[t+1] = (1+rate)*s[t] + wage - c[t];

    retired {t in R..T}:
        s[t+1] = (1+rate)*s[t] - c[t];

initial:
    s[0] = 0;

terminal:
    s[T+1] = 0;
Data: life.dat

param R := 75;  # Retirement age
param T := 100;  # Terminal age
param beta := 0.9;  # Discount factor
param rate := 0.2;  # Interest rate
param wage := 1.0;  # Wage rate
# Load model and data
model life1.mod;
data life.dat;

# Specify solver and options
option solver mpec;

# Solve the instance
solve;

# Output results
printf {t in 0..T} "%2d %5.4e %5.4e\n", t, s[t], c[t] > out1.dat;
Output

ampl: include life1.cmd
AMPL interface to filter-MPEC: 20040408
: filter objective function  = -3.24322
   constraint violation = 1.01433e-11
Optimal solution found
14 iterations (0 for feasibility)
Evals: obj = 15, constr = 16, grad = 16, Hes = 15
ampl: quit;
Plot of Output
Model: life2.mod

param R > 0, integer;  # Retirement age
param T > R, integer;  # Terminal age

param beta >= 0, < 1;  # Discount factor
param rate >= 0, < 1;  # Interest rate
param wage >= 0;  # Wage rate

var cbar{0..T};  # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t;  # Actual consumption
var s{0..T+1};  # Savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
   sum {t in 0..T} beta^t * u[t];

subject to
   working {t in 0..R-1}:
      s[t+1] = (1+rate)*s[t] + wage - cbar[t] / beta^t;

   retired {t in R..T}:
      s[t+1] = (1+rate)*s[t] - cbar[t] / beta^t;

initial:
   s[0] = 0;

terminal:
   s[T+1] = 0;
Plot of Output
Model: life3.mod

param R > 0, integer; # Retirement age
param T > R, integer; # Terminal age
param beta >= 0, < 1; # Discount factor
param rate >= 0, < 1; # Interest rate
param wage >= 0; # Wage rate

var cbar{0..T}; # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t; # Actual consumption
var s{0..T+1}; # Savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
    working {t in 0..R-1}:
        beta^t*s[t+1] = beta^t*(1+rate)*s[t] + beta^t*wage - cbar[t];

    retired {t in R..T}:
        beta^t*s[t+1] = beta^t*(1+rate)*s[t] - cbar[t];

initial:
    s[0] = 0;

terminal:
    s[T+1] = 0;
Plot of Output
Model: life4.mod

param R > 0, integer; # Retirement age
param T > R, integer; # Terminal age

param beta >= 0, < 1; # Discount factor
param rate >= 0, < 1; # Interest rate
param wage >= 0; # Wage rate

var cbar{0..T}; # Scaled consumption
var c{t in 0..T} = cbar[t] / beta^t; # Actual consumption
var sbar{0..T+1}; # Scaled savings
var s{t in 0..T+1} = sbar[t] / beta^t; # Actual savings
var u{t in 0..T} = -exp(-cbar[t] / beta^t);

maximize utility:
    sum {t in 0..T} beta^t * u[t];

subject to
    working {t in 0..R-1}:
        sbar[t+1]/beta = (1+rate)*sbar[t] + beta^t*wage - cbar[t];
    retired {t in R..T}:
        sbar[t+1]/beta = (1+rate)*sbar[t] - cbar[t];

initial:
    sbar[0] = 0;

terminal:
    sbar[T+1] = 0;
Plot of Output
Solving Constrained Optimization Problems

\[
\min_x f(x) \quad \text{subject to} \quad c(x) \geq 0
\]

Main ingredients of solution approaches:
- **Local method**: given \( x_k \) (solution guess) find a step \( s \).
  - Sequential Quadratic Programming (SQP)
  - Sequential Linear/Quadratic Programming (SLQP)
  - Interior-Point Method (IPM)
- **Globalization strategy**: converge from any starting point.
  - Trust region
  - Line search
- **Acceptance criteria**: filter or penalty function.
Interior-Point Method

- Reformulate optimization problem with slacks

\[
\begin{align*}
\min_{x} & \quad f(x) \\
\text{subject to} & \quad c(x) = 0 \\
& \quad x \geq 0
\end{align*}
\]

- Construct perturbed optimality conditions

\[
F_\tau(x, y, z) = \begin{bmatrix}
\nabla f(x) - \nabla c(x)^T y - z \\
c(x) \\
Xz - \tau e
\end{bmatrix}
\]

- Central path \( \{x(\tau), y(\tau), z(\tau) \mid \tau > 0\} \)
- Apply Newton’s method for sequence \( \tau \downarrow 0 \)
Interior-Point Method

1. Compute a new iterate
   1.1 Solve linear system of equations

\[
\begin{bmatrix}
W_k & -\nabla c(x_k)^T & -I \\
\nabla c(x_k) & 0 & 0 \\
Z_k & 0 & X_k
\end{bmatrix}
\begin{pmatrix}
s_x \\
s_y \\
s_z
\end{pmatrix}
= -F_\mu(x_k, y_k, z_k)
\]

1.2 Accept or reject iterate
1.3 Update parameters

2. Check convergence
Convergence Issues

- Quadratic convergence – best outcome
- Globally infeasible – linear constraints infeasible
- Locally infeasible – nonlinear constraints locally infeasible
- Dual infeasible – dual problem is locally infeasible
- Unbounded objective – hard to detect
- Unbounded multipliers – constraint qualification not satisfied
- Duality gap
- Domain violations such as $\frac{1}{x}$ when $x = 0$
  - Make implicit constraints explicit
- Nonglobal solutions
  - Apply a multistart heuristic
  - Use global optimization solver
Some Available Software

- IPOPT – open source in COIN-OR
  - line-search filter algorithm
- KNITRO
  - trust-region Newton to solve barrier problem
  - $\ell_1$ penalty barrier function
  - Newton system: direct solves or null-space CG
- LOQO
  - line-search method
  - Newton system: modified Cholesky factorization
Optimal Technology

Optimize energy production schedule and transition between old and new reduced-carbon technology to meet carbon targets

• Maximize social welfare

• Constraints
  • Limit total greenhouse gas emissions
  • Low-carbon technology less costly as it becomes widespread

• Assumptions on emission rates, economic growth, and energy costs
Model Formulation

- Finite time: \( t \in [0, T] \)
- Instantaneous energy output: \( q^o(t) \) and \( q^n(t) \)
- Cumulative energy output: \( x^o(t) \) and \( x^n(t) \)

\[
x^n(t) = \int_0^t q^n(\tau) \, d\tau
\]

- Discounted greenhouse gases emissions

\[
\int_0^T e^{-at} \left( b_o q^o(t) + b_n q^n(t) \right) \, dt \leq z_T
\]

- Consumer surplus \( S(Q(t), t) \) derived from utility

- Production costs
  - \( c_o \) per unit cost of old technology
  - \( c_n(x^n(t)) \) per unit cost of new technology (learning by doing)
Continuous-Time Model

\[
\max \{q^o, q^n, x^n, z\}(t)
\]

\[
\int_0^T e^{-rt} \left[ S(q^o(t) + q^n(t), t) - c_0 q^o(t) - c_n(x^n(t))q^n(t) \right] dt
\]

subject to

\[
x^n(t) = q^n(t) \quad x(0) = x_0 = 0
\]

\[
\dot{z}(t) = e^{-at} (b_0 q^o(t) + b_n q^n(t)) \quad z(0) = z_0 = 0
\]

\[
z(T) \leq z_T
\]

\[
q^o(t) \geq 0, \quad q^n(t) \geq 0.
\]
Optimal Technology Penetration

Discretization:

- $t \in [0, T]$ replaced by $N + 1$ equally spaced points $t_i = ih$
- $h := T/N$ time integration step-length
- approximate $q_i^n \simeq q^n(t_i)$ etc.

Replace differential equation

$$\dot{x}(t) = q^n(t)$$

by

$$x_{i+1} = x_i + h q_i^n$$
Optimal Technology Penetration

Discretization:

- $t \in [0, T]$ replaced by $N + 1$ equally spaced points $t_i = ih$
- $h := T/N$ time integration step-length
- approximate $q^n_i \simeq q^n(t_i)$ etc.

Replace differential equation

$$\dot{x}(t) = q^n(t)$$

by

$$x_{i+1} = x_i + hq^n_i$$

Output of new technology between $t = 24$ and $t = 35$
Solution with Varying $h$

Output for different discretization schemes and step-sizes
Optimal Technology Penetration

Add adjustment cost to model building of capacity:

**Capital and Investment:**

- $K^j(t)$ amount of capital in technology $j$ at $t$.
- $I^j(t)$ investment to increase $K^j(t)$.
- initial capital level as $\bar{K}^j_0$:

**Notation:**

- $Q(t) = q^o(t) + q^n(t)$
- $C(t) = C^o(q^o(t), K^o(t)) + C^n(q^n(t), K^n(t))$
- $I(t) = I^o(t) + I^n(t)$
- $K(t) = K^o(t) + K^n(t)$
Optimal Technology Penetration

\[
\text{maximize } \left\{ \int_0^T e^{-rt} \left[ \tilde{S}(Q(t), t) - C(t) - K(t) \right] dt + e^{-rT}K(T) \right\}
\]

subject to

\[
\begin{align*}
\dot{x}(t) &= q^n(t), \quad x(0) = x_0 = 0 \\
\dot{K}^j(t) &= -\delta K^j(t) + I^j(t), \quad K^j(0) = \bar{K}^j_0, \quad j \in \{o, n\} \\
\dot{z}(t) &= e^{-at}[b_o q^o(t) + b_n q^n(t)], \quad z(0) = z_0 = 0 \\
z(T) &\leq z_T \\
q^j(t) &\geq 0, \quad j \in \{o, n\} \\
I^j(t) &\geq 0, \quad j \in \{o, n\}
\end{align*}
\]
Optimal Technology Penetration

Optimal output, investment, and capital for 50% CO2 reduction.
Pitfalls of Discretizations [Hager, 2000]

Optimal Control Problem

\[
\text{minimize } \frac{1}{2} \int_0^1 u^2(t) + 2y^2(t) dt
\]

subject to

\[
\dot{y}(t) = \frac{1}{2} y(t) + u(t), \quad t \in [0, 1],
\]
\[
y(0) = 1.
\]

\[
\Rightarrow y^*(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)},
\]
\[
u^*(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}.
\]
Pitfalls of Discretizations [Hager, 2000]

Optimal Control Problem

\[ \begin{align*}
\text{minimize} & \quad \frac{1}{2} \int_{0}^{1} u^2(t) + 2y^2(t) \, dt \\
\text{subject to} & \quad \dot{y}(t) = \frac{1}{2} y(t) + u(t), \ t \in [0, 1], \\
y(0) & = 1.
\end{align*} \]

\[ \Rightarrow y^*(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \]

\[ u^*(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}. \]

Discretize with 2nd order RK

\[ \begin{align*}
\text{minimize} & \quad \frac{h}{2} \sum_{k=0}^{K-1} u_{k+1/2}^2 + 2y_{k+1/2}^2 \\
\text{subject to} & \quad (k = 0, \ldots, K):
\end{align*} \]

\[ \begin{align*}
y_{k+1/2} & = y_k + \frac{h}{2} \left( \frac{1}{2} y_k + u_k \right), \\
y_{k+1} & = y_k + h \left( \frac{1}{2} y_{k+1/2} + u_{k+1/2} \right).
\end{align*} \]
Pitfalls of Discretizations [Hager, 2000]

Optimal Control Problem

\[
\begin{align*}
\text{minimize} & \quad \frac{1}{2} \int_0^1 u^2(t) + 2y^2(t) \, dt \\
\text{subject to} & \quad \dot{y}(t) = \frac{1}{2} y(t) + u(t), \quad t \in [0, 1], \\
& \quad y(0) = 1.
\end{align*}
\]

\[
\Rightarrow y^*(t) = \frac{2e^{3t} + e^3}{e^{3t/2}(2 + e^3)}, \\
u^*(t) = \frac{2(e^{3t} - e^3)}{e^{3t/2}(2 + e^3)}.
\]

Discretize with 2nd order RK

\[
\begin{align*}
\text{minimize} & \quad \frac{h}{2} \sum_{k=0}^{K-1} u_{k+1/2}^2 + 2y_{k+1/2}^2 \\
\text{subject to} & \quad (k = 0, \ldots, K): \\
& \quad y_{k+1/2} = y_k + \frac{h}{2} \left( \frac{1}{2} y_k + u_k \right), \\
& \quad y_{k+1} = y_k + h \left( \frac{1}{2} y_{k+1/2} + u_{k+1/2} \right).
\end{align*}
\]

Discrete solution \((k = 0, \ldots, K)\):

\[
\begin{align*}
y_k & = 1, \quad y_{k+1/2} = 0, \\
u_k & = -\frac{4 + h}{2h}, \quad u_{k+1/2} = 0,
\end{align*}
\]
Tips to Solve Continuous-Time Problems

- Use discretize-then-optimize with different schemes
- Refine discretization: $h = 1$ discretization is nonsense
- Check implied discretization of adjoints
Tips to Solve Continuous-Time Problems

- Use discretize-then-optimize with different schemes
- Refine discretization: $h = 1$ discretization is nonsense
- Check implied discretization of adjoints

Alternative: Optimize-Then-Discretize
- Consistent adjoint/dual discretization
- Discretized gradients can be wrong!
- Harder for inequality constraints
Ordered Sets

param V, integer; # Number of vertices
param E, integer; # Number of elements

set VERTICES := {1..V}; # Vertex indices
set ELEMENTS := {1..E}; # Element indices
set COORDS := {1..3} ordered; # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES; # Tetrahedral elements

var x{VERTICES, COORDS}; # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
                         (x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
                         (x[T[e,2], i] - x[T[e,1], i]) *
                         ((x[T[e,3], nextw(i)] - x[T[e,1], nextw(i)]) *
                          (x[T[e,4], prevw(i)] - x[T[e,1], prevw(i)]) -
                          (x[T[e,3], prevw(i)] - x[T[e,1], prevw(i)]) *
                          (x[T[e,4], nextw(i)] - x[T[e,1], nextw(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
Circular Sets

param V, integer;                  # Number of vertices
param E, integer;                  # Number of elements

set VERTICES := {1..V};            # Vertex indices
set ELEMENTS := {1..E};            # Element indices
set COORDS := {1..3} circular;     # Spatial coordinates

param T{ELEMENTS, 1..4} in VERTICES; # Tetrahedral elements

var x{VERTICES, COORDS};           # Position of vertices

var norm{e in ELEMENTS} = sum{i in COORDS, j in 1..4}
(x[T[e,j], i] - x[T[e,1], i])^2;

var area{e in ELEMENTS} = sum{i in COORDS}
(x[T[e,2], i] - x[T[e,1], i]) *
((x[T[e,3], next(i)] - x[T[e,1], next(i)]) *
 (x[T[e,4], prev(i)] - x[T[e,1], prev(i)]) -
 (x[T[e,3], prev(i)] - x[T[e,1], prev(i)]) *
 (x[T[e,4], next(i)] - x[T[e,1], next(i)]));

minimize f: sum {e in ELEMENTS} norm[e] / max(area[e], 0) ^ (2 / 3);
Part IV

Numerical Optimization III: Complementarity Constraints
Nash Games

- Non-cooperative game played by \( n \) individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible
Nash Games

- Non-cooperative game played by \( n \) individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible
- Characterization of two player equilibrium \( (x^*, y^*) \)

\[
\begin{align*}
x^* & \in \left\{ \arg \min_{x \geq 0} f_1(x, y^*) \right. \\
& \quad \text{subject to } c_1(x) \leq 0 \\
y^* & \in \left\{ \arg \min_{y \geq 0} f_2(x^*, y) \right. \\
& \quad \text{subject to } c_2(y) \leq 0
\end{align*}
\]
Nash Games

- Non-cooperative game played by $n$ individuals
  - Each player selects a strategy to optimize their objective
  - Strategies for the other players are fixed
- Equilibrium reached when no improvement is possible
- Characterization of two player equilibrium $(x^*, y^*)$

$$\begin{align*}
x^* \in & \left\{ \arg \min_{x \geq 0} f_1(x, y^*) \right. \\
& \text{subject to } c_1(x) \leq 0 \\
y^* \in & \left\{ \arg \min_{y \geq 0} f_2(x^*, y) \right. \\
& \text{subject to } c_2(y) \leq 0
\end{align*}$$

- Many applications in economics
  - Bimatrix games
  - Cournot duopoly models
  - General equilibrium models
  - Arrow-Debreu models
Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each $y$
  - $f_2(x, \cdot)$ is convex for each $x$
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification
- Then the first-order conditions are necessary and sufficient

$$\begin{align*}
\min_{x \geq 0} & \quad f_1(x, y^*) \\
\text{subject to} & \quad c_1(x) \leq 0
\end{align*}$$

$$\Leftrightarrow \begin{align*}
0 \leq x & \perp \nabla_x f_1(x, y^*) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\
0 \leq \lambda_1 & \perp -c_1(x) \geq 0
\end{align*}$$
Complementarity Formulation

• Assume each optimization problem is convex
  • $f_1(\cdot, y)$ is convex for each $y$
  • $f_2(x, \cdot)$ is convex for each $x$
  • $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification

• Then the first-order conditions are necessary and sufficient

\[
\begin{align*}
\min_{y \geq 0} & \quad f_2(x^*, y) \\
\text{subject to} & \quad c_2(y) \leq 0
\end{align*}
\quad \Leftrightarrow \quad \begin{cases} 
0 \leq y & \perp \nabla_y f_2(x^*, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\
0 \leq \lambda_2 & \perp -c_2(y) \geq 0
\end{cases}
\]
Complementarity Formulation

- Assume each optimization problem is convex
  - $f_1(\cdot, y)$ is convex for each $y$
  - $f_2(x, \cdot)$ is convex for each $x$
  - $c_1(\cdot)$ and $c_2(\cdot)$ satisfy constraint qualification

- Then the first-order conditions are necessary and sufficient

\[
\begin{align*}
0 \leq x & \perp \nabla_x f_1(x, y) + \lambda_1^T \nabla_x c_1(x) \geq 0 \\
0 \leq y & \perp \nabla_y f_2(x, y) + \lambda_2^T \nabla_y c_2(y) \geq 0 \\
0 \leq \lambda_1 & \perp -c_1(y) \geq 0 \\
0 \leq \lambda_2 & \perp -c_2(y) \geq 0
\end{align*}
\]

- Nonlinear complementarity problem
  - Square system – number of variables and constraints the same
  - Each solution is an equilibrium for the Nash game
Model Formulation

- Economy with $n$ agents and $m$ commodities
  - $e \in \mathbb{R}^{n \times m}$ are the endowments
  - $\alpha \in \mathbb{R}^{n \times m}$ and $\beta \in \mathbb{R}^{n \times m}$ are the utility parameters
  - $p \in \mathbb{R}^m$ are the commodity prices
- Agent $i$ maximizes utility with budget constraint

$$
\max_{x_{i,*} \geq 0} \sum_{k=1}^{m} \frac{\alpha_{i,k}(1 + x_{i,k})^{1-\beta_{i,k}}}{1 - \beta_{i,k}}
$$

subject to

$$
\sum_{k=1}^{m} p_k (x_{i,k} - e_{i,k}) \leq 0
$$

- Market $k$ sets price for the commodity

$$
0 \leq p_k \perp \sum_{i=1}^{n} (e_{i,k} - x_{i,k}) \geq 0
$$
Model: cge.mod

```plaintext
set AGENTS;                     # Agents
set COMMODITIES;                # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment
param alpha {AGENTS, COMMODITIES} > 0; # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES};    # Consumption (no bounds!)
var l {AGENTS};                 # Multipliers (no bounds!)
var p {COMMODITIES};            # Prices (no bounds!)

var du {i in AGENTS, k in COMMODITIES} =
    alpha[i,k] / (1 + x[i,k])^beta[i,k]; # Marginal prices

subject to
    optimality {i in AGENTS, k in COMMODITIES}:
        0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

    budget {i in AGENTS}:
        0 <= l[i] complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

    market {k in COMMODITIES}:
        0 <= p[k] complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
```
Data: cge.dat

set AGENTS := Jorge, Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books Cars Food Pens :=
          Jorge   1   1   1   1
          Sven    1   2   3   4
          Todd    2   1   1   5;

param beta (tr): Jorge Sven Todd :=
       Books  1.5  2   0.6
       Cars   1.6  3   0.7
       Food   1.7  2   2.0
       Pens   1.8  2   2.5;
Commands: cge.cmd

# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cge.out;
printf "\n" > cge.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cge.out;
Results: cge.out

Jorge Books: 8.9825e-01
Jorge Cars: 1.4651e+00
Jorge Food: 1.2021e+00
Jorge Pens: 6.8392e-01
Sven Books: 2.5392e-01
Sven Cars: 7.2054e-01
Sven Food: 1.6271e+00
Sven Pens: 1.4787e+00
Todd Books: 1.8478e+00
Todd Cars: 8.1431e-01
Todd Food: 1.7081e-01
Todd Pens: 8.3738e-01

Books: 1.0825e+01
Cars: 6.6835e+00
Food: 7.3983e+00
Pens: 1.1081e+01
Commands: cgenum.cmd

# Load model and data
model cge.mod;
data cge.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgenum.out;
printf "\n" > cgenum.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgenum.out;
Results: cgenum.out

Jorge Books: 8.9825e-01
Jorge Cars: 1.4651e+00
Jorge Food: 1.2021e+00
Jorge Pens: 6.8392e-01
Sven Books: 2.5392e-01
Sven Cars: 7.2054e-01
Sven Food: 1.6271e+00
Sven Pens: 1.4787e+00
Todd Books: 1.8478e+00
Todd Cars: 8.1431e-01
Todd Food: 1.7081e-01
Todd Pens: 8.3738e-01

Books: 1.0000e+00
Cars: 6.1742e-01
Food: 6.8345e-01
Pens: 1.0237e+00
Pitfalls

- Nonsquare systems
  - Side variables
  - Side constraints
- Orientation of equations
  - Skew symmetry preferred
  - Proximal point perturbation
- AMPL presolve
  - option presolve 0;
Newton Method for Nonlinear Equations
Newton Method for Nonlinear Equations

![Graph showing Newton's method for nonlinear equations.](image-url)
Newton Method for Nonlinear Equations
Newton Method for Nonlinear Equations

\[ F(x) \]
Methods for Complementarity Problems

- **Sequential linearization methods (PATH)**
  1. Solve the linear complementarity problem
     
     \[ 0 \leq x \perp F(x_k) + \nabla F(x_k)(x - x_k) \geq 0 \]
  2. Perform a line search along merit function
  3. Repeat until convergence
Methods for Complementarity Problems

- Sequential linearization methods (PATH)
  1. Solve the linear complementarity problem
     \[ 0 \leq x \perp F(x_k) + \nabla F(x_k)(x - x_k) \geq 0 \]
  2. Perform a line search along merit function
  3. Repeat until convergence

- Semismooth reformulation methods (SEMI)
  - Solve linear system of equations to obtain direction
  - Globalize with a trust region or line search
  - Less robust in general

- Interior-point methods
Semismooth Reformulation

• Define Fischer-Burmeister function

\[ \phi(a, b) := a + b - \sqrt{a^2 + b^2} \]

\[ \phi(a, b) = 0 \text{ iff } a \geq 0, \ b \geq 0, \text{ and } ab = 0 \]

• Define the system

\[ [\Phi(x)]_i = \phi(x_i, F_i(x)) \]

• \( x^* \) solves complementarity problem iff \( \Phi(x^*) = 0 \)

• Nonsmooth system of equations
Semismooth Algorithm

1. Calculate $H^k \in \partial_B \Phi(x^k)$ and solve the following system for $d^k$:

$$H^k d^k = -\Phi(x^k)$$

If this system either has no solution, or

$$\nabla \Psi(x^k)^T d^k \leq -p_1 \|d^k\|^{p_2}$$

is not satisfied, let $d^k = -\nabla \Psi(x^k)$.
Semismooth Algorithm

1. Calculate \( H^k \in \partial_B \Phi(x^k) \) and solve the following system for \( d^k \):

\[
H^k d^k = -\Phi(x^k)
\]

If this system either has no solution, or

\[
\nabla \Psi(x^k)^T d^k \leq -p_1 \|d^k\|^{p_2}
\]

is not satisfied, let \( d^k = -\nabla \Psi(x^k) \).

2. Compute smallest nonnegative integer \( i^k \) such that

\[
\Psi(x^k + \beta i^k d^k) \leq \Psi(x^k) + \sigma \beta i^k \nabla \Psi(x^k) d^k
\]

3. Set \( x^{k+1} = x^k + \beta i^k d^k \), \( k = k + 1 \), and go to 1.
Convergence Issues

- Quadratic convergence – best outcome
- Linear convergence
  - Far from a solution – $r(x_k)$ is large
  - Jacobian is incorrect – disrupts quadratic convergence
  - Jacobian is rank deficient – $\|\nabla r(x_k)\|$ is small
  - Converge to local minimizer – guarantees rank deficiency
  - Limits of finite precision arithmetic
    1. $r(x_k)$ converges quadratically to small number
    2. $r(x_k)$ hovers around that number with no progress
- Domain violations such as $\frac{1}{x}$ when $x = 0$
Some Available Software

- PATH – sequential linearization method
- MILES – sequential linearization method
- SEMI – semismooth linesearch method
- TAO – Toolkit for Advanced Optimization
  - SSLS – full-space semismooth linesearch methods
  - ASLS – active-set semismooth linesearch methods
  - RSCS – reduced-space method
Definition

- Leader-follower game
  - Dominant player (leader) selects a strategy $y^*$
  - Then followers respond by playing a Nash game

$$x^*_i \in \left\{ \begin{array}{l} \arg \min_{x_i \geq 0} f_i(x, y) \\ \text{subject to } c_i(x_i) \leq 0 \end{array} \right\}$$

- Leader solves optimization problem with equilibrium constraints

$$\min_{y \geq 0, x, \lambda} g(x, y)$$
subject to

$$h(y) \leq 0$$
$$0 \leq x_i \perp \nabla x_i f_i(x, y) + \lambda_i^T \nabla x_i c_i(x_i) \geq 0$$
$$0 \leq \lambda_i \perp -c_i(x_i) \geq 0$$

- Many applications in economics
  - Optimal taxation
  - Tolling problems
Model Formulation

• Economy with $n$ agents and $m$ commodities
  • $e \in \mathbb{R}^{n \times m}$ are the endowments
  • $\alpha \in \mathbb{R}^{n \times m}$ and $\beta \in \mathbb{R}^{n \times m}$ are the utility parameters
  • $p \in \mathbb{R}^m$ are the commodity prices
• Agent $i$ maximizes utility with budget constraint

$$
\max_{x_i, \star \geq 0} \sum_{k=1}^{m} \frac{\alpha_i,k (1 + x_i,k)^{1-\beta_i,k}}{1 - \beta_i,k}
$$
subject to

$$
\sum_{k=1}^{m} p_k (x_i,k - e_i,k) \leq 0
$$

• Market $k$ sets price for the commodity

$$
0 \leq p_k \perp \sum_{i=1}^{n} (e_i,k - x_i,k) \geq 0
$$
Model: cgempec.mod

set LEADER; # Leader
set FOLLOWERS; # Followers
set AGENTS := LEADER union FOLLOWERS; # All the agents
cHECK: (card(LEADER) == 1 && card(LEADER inter FOLLOWERS) == 0);

set COMMODITIES; # Commodities

param e {AGENTS, COMMODITIES} >= 0, default 1; # Endowment

param alpha {AGENTS, COMMODITIES} > 0; # Utility parameters
param beta {AGENTS, COMMODITIES} > 0;

var x {AGENTS, COMMODITIES}; # Consumption (no bounds!)
var l {FOLLOWERS}; # Multipliers (no bounds!)
var p {COMMODITIES}; # Prices (no bounds!)

var u {i in AGENTS} =
    sum {k in COMMODITIES} alpha[i,k] * (1 + x[i,k])^(1 - beta[i,k]) / (1 - beta[i,k]);
var du {i in AGENTS, k in COMMODITIES} =
    alpha[i,k] / (1 + x[i,k])^beta[i,k]; # Marginal prices
Model: cgempec.mod

maximize
  objective: sum {i in LEADER} u[i];

subject to
  leader_budget {i in LEADER}:
    sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  optimality {i in FOLLOWERS, k in COMMODITIES}:
    0 <= x[i,k] complements -du[i,k] + p[k] * l[i] >= 0;

  budget {i in FOLLOWERS}:
    0 <= l[i] complements sum {k in COMMODITIES} p[k]*(e[i,k] - x[i,k]) >= 0;

  market {k in COMMODITIES}:
    0 <= p[k] complements sum {i in AGENTS} (e[i,k] - x[i,k]) >= 0;
Data: cgempec.dat

```plaintext
set LEADER := Jorge;
set FOLLOWERS := Sven, Todd;
set COMMODITIES := Books, Cars, Food, Pens;

param alpha : Books Cars Food Pens :=
    Jorge 1 1 1 1
    Sven 1 2 3 4
    Todd 2 1 1 5;

param beta (tr): Jorge Sven Todd :=
    Books 1.5 2 0.6
    Cars 1.6 3 0.7
    Food 1.7 2 2.0
    Pens 1.8 2 2.5;
```

# Load model and data
model cgempec.mod;
data cgempec.dat;

# Specify solver and options
option presolve 0;
option solver "loqo";

# Solve the instance
drop market['Books'];
fix p['Books'] := 1;
solve;

# Output results
printf {i in AGENTS, k in COMMODITIES} "%5s %5s: % 5.4e\n", i, k, x[i,k] > cgempec.out;
printf "\n" > cgempec.out;
printf {k in COMMODITIES} "%5s: % 5.4e\n", k, p[k] > cgempec.out;
Output: cgempec.out

<table>
<thead>
<tr>
<th></th>
<th>Stackleberg</th>
<th>Nash Game</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jorge Books</td>
<td>9.2452e-01</td>
<td>Jorge Books:</td>
</tr>
<tr>
<td>Jorge Cars</td>
<td>1.3666e+00</td>
<td>1.4651e+00</td>
</tr>
<tr>
<td>Jorge Food</td>
<td>1.1508e+00</td>
<td>1.2021e+00</td>
</tr>
<tr>
<td>Jorge Pens</td>
<td>7.7259e-01</td>
<td>6.8392e-01</td>
</tr>
<tr>
<td>Sven Books</td>
<td>2.5499e-01</td>
<td>2.5392e-01</td>
</tr>
<tr>
<td>Sven Cars</td>
<td>7.4173e-01</td>
<td>7.2054e-01</td>
</tr>
<tr>
<td>Sven Food</td>
<td>1.6657e+00</td>
<td>1.6271e+00</td>
</tr>
<tr>
<td>Sven Pens</td>
<td>1.4265e+00</td>
<td>1.4787e+00</td>
</tr>
<tr>
<td>Todd Books</td>
<td>1.8205e+00</td>
<td>1.8478e+00</td>
</tr>
<tr>
<td>Todd Cars</td>
<td>8.9169e-01</td>
<td>8.1431e-01</td>
</tr>
<tr>
<td>Todd Food</td>
<td>1.8355e-01</td>
<td>1.7081e-01</td>
</tr>
<tr>
<td>Todd Pens</td>
<td>8.0093e-01</td>
<td>8.3738e-01</td>
</tr>
</tbody>
</table>

Books: 1.0000e+00   Books: 1.0000e+00
Cars: 5.9617e-01    Cars: 6.1742e-01
Food: 6.6496e-01   Food: 6.8345e-01
Pens: 1.0700e+00   Pens: 1.0237e+00
Nonlinear Programming Formulation

\[
\begin{align*}
\min_{x,y,\lambda,s,t \geq 0} & \quad g(x, y) \\
\text{subject to} & \quad h(y) \leq 0 \\
& \quad s_i = -c_i(x_i) \\
& \quad t_i = -c_i(x_i) \\
& \quad \sum_i (s_i^T x_i + \lambda_i t_i) \leq 0
\end{align*}
\]

- Constraint qualification fails
  - Lagrange multiplier set unbounded
  - Constraint gradients linearly dependent
  - Central path does not exist
- Able to prove convergence results for some methods
- Reformulation very successful and versatile in practice
Penalization Approach

\[
\begin{align*}
\min_{x,y,\lambda,s,t \geq 0} & \quad g(x, y) + \pi \sum_i \left( s_i^T x_i + \lambda_i t_i \right) \\
\text{subject to} & \quad h(y) \leq 0 \\
& \quad s_i = \nabla_{x_i} f_i(x, y) + \lambda_i^T \nabla_{x_i} c_i(x_i) \\
& \quad t_i = -c_i(x_i)
\end{align*}
\]

- Optimization problem satisfies constraint qualification
- Need to increase \( \pi \)


Relaxation Approach

\[
\begin{align*}
\min_{x, y, \lambda, s, t \geq 0} & \quad g(x, y) \\
\text{subject to} & \quad h(y) \leq 0 \\
& \quad s_i = \nabla x f_i(x, y) + \lambda^T_i \nabla x c_i(x_i) \\
& \quad t_i = -c_i(x_i) \\
& \quad \sum_i (s_i^T x_i + \lambda_i t_i) \leq \tau
\end{align*}
\]

- Need to decrease \( \tau \)
Limitations

- Multipliers may not exist
- Solvers can have a hard time computing solutions
  - Try different algorithms
  - Compute feasible starting point
- Stationary points may have descent directions
  - Checking for descent is an exponential problem
  - Strong stationary points found in certain cases
- Many stationary points – global optimization
Limitations

- Multipliers may not exist
- Solvers can have a hard time computing solutions
  - Try different algorithms
  - Compute feasible starting point
- Stationary points may have descent directions
  - Checking for descent is an exponential problem
  - Strong stationary points found in certain cases
- Many stationary points – global optimization
- Formulation of follower problem
  - Multiple solutions to Nash game
  - Nonconvex objective or constraints
  - Existence of multipliers
Model Formulation

- Firm $f \in F$ chooses output $x_f$ to maximize profit
  - $u$ is the utility function
  $$u = \left( 1 + \sum_{f \in F} x_f^\alpha \right)^{\frac{\eta}{\alpha}}$$
  - $\alpha$ and $\eta$ are parameters
  - $c_f$ is the unit cost for each firm
- In particular, for each firm $f \in F$
  $$x_f^* \in \arg \max_{x_f \geq 0} \left( \frac{\partial u}{\partial x_f} - c_f \right) x_f$$
- First-order optimality conditions
  $$0 \leq x_f \perp c_f - \frac{\partial u}{\partial x_f} - x_f \frac{\partial^2 u}{\partial x_f^2} \geq 0$$
Model: oligopoly.mod

set FIRMS; # Firms in problem

param c {FIRMS}; # Unit cost
param alpha > 0; # Constants
param eta > 0;

var x {FIRMS} default 0.1; # Output (no bounds!)

var s = 1 + sum {f in FIRMS} x[f]^alpha; # Summation term
var u = s^(eta/alpha); # Utility
var du {f in FIRMS} = # Marginal price
teta * s^(eta/alpha - 1) * x[f]^(alpha - 1);
var dudu {f in FIRMS} = # Derivative
teta * (eta - alpha) * s^(eta/alpha - 2) * x[f]^(2 * alpha - 2) +
eteta * (alpha - 1) * s^(eta/alpha - 1) * x[f]^(alpha - 2);

compl {f in FIRMS}:
0 <= x[f] complements c[f] - du[f] - x[f] * dudu[f] >= 0;
Data: oligopoly.dat

param: FIRMS : c :=
  1  0.07
  2  0.08
  3  0.09;

param alpha := 0.999;
param eta := 0.2;
Commands: oligopoly.cmd

# Load model and data
model oligopoly.mod;
data oligopoly.dat;

# Specify solver and options
option presolve 0;
option solver "pathampl";

# Solve complementarity problem
solve;

# Output the results
printf {f in FIRMS} "Output for firm %2d: % 5.4e\n", f, x[f] > oligcomp.out;
Results: oligopoly.out

Output for firm 1: 8.3735e-01
Output for firm 2: 5.0720e-01
Output for firm 3: 1.7921e-01
Model Formulation

- Players select strategies to minimize loss
  - $p \in \mathbb{R}^n$ is the probability player 1 chooses each strategy
  - $q \in \mathbb{R}^m$ is the probability player 2 chooses each strategy
  - $A \in \mathbb{R}^{n \times m}$ is the loss matrix for player 1
  - $B \in \mathbb{R}^{n \times m}$ is the loss matrix for player 2

- Optimization problem for player 1

  $$\begin{align*}
  \min_{0 \leq p \leq 1} & \quad p^T A q \\
  \text{subject to} & \quad e^T p = 1
  \end{align*}$$

- Optimization problem for player 2

  $$\begin{align*}
  \min_{0 \leq q \leq 1} & \quad p^T B q \\
  \text{subject to} & \quad e^T q = 1
  \end{align*}$$
Model Formulation

- Players select strategies to minimize loss
  - \( p \in \mathbb{R}^n \) is the probability player 1 chooses each strategy
  - \( q \in \mathbb{R}^m \) is the probability player 2 chooses each strategy
  - \( A \in \mathbb{R}^{n \times m} \) is the loss matrix for player 1
  - \( B \in \mathbb{R}^{n \times m} \) is the loss matrix for player 2

- Complementarity problem

\[
\begin{align*}
0 \leq p & \leq 1 \ Perp Aq - \lambda_1 \\
0 \leq q & \leq 1 \ Perp B^T p - \lambda_2 \\
\lambda_1 & \text{ free} \ Perp e^T p = 1 \\
\lambda_2 & \text{ free} \ Perp e^T q = 1
\end{align*}
\]
Model: bimatrix1.mod

param n > 0, integer;  # Strategies for player 1
param m > 0, integer;  # Strategies for player 2

param A{1..n, 1..m};  # Loss matrix for player 1
param B{1..n, 1..m};  # Loss matrix for player 2

var p{1..n};  # Probability player 1 selects strategy i
var q{1..m};  # Probability player 2 selects strategy j
var lambda1;  # Multiplier for constraint
var lambda2;  # Multiplier for constraint

subject to
  opt1 {i in 1..n}:  # Optimality conditions for player 1
    0 <= p[i] <= 1 complements sum{j in 1..m} A[i,j] * q[j] - lambda1;

  opt2 {j in 1..m}:  # Optimality conditions for player 2
    0 <= q[j] <= 1 complements sum{i in 1..n} B[i,j] * p[i] - lambda2;

  con1:
    lambda1 complements sum{i in 1..n} p[i] = 1;

  con2:
    lambda2 complements sum{j in 1..m} q[j] = 1;
Model: bimatrix2.mod

param n > 0, integer; # Strategies for player 1
param m > 0, integer; # Strategies for player 2

param A{1..n, 1..m}; # Loss matrix for player 1
param B{1..n, 1..m}; # Loss matrix for player 2

var p{1..n}; # Probability player 1 selects strategy i
var q{1..m}; # Probability player 2 selects strategy j
var lambda1; # Multiplier for constraint
var lambda2; # Multiplier for constraint

subject to
  opt1 {i in 1..n}: # Optimality conditions for player 1
  0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] - lambda1 >= 0;

  opt2 {j in 1..m}: # Optimality conditions for player 2
  0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] - lambda2 >= 0;

  con1:
  0 <= lambda1 complements sum{i in 1..n} p[i] >= 1;

  con2:
  0 <= lambda2 complements sum{j in 1..m} q[j] >= 1;
Model: bimatrix3.mod

param n > 0, integer;           # Strategies for player 1
param m > 0, integer;           # Strategies for player 2

param A{1..n, 1..m};           # Loss matrix for player 1
param B{1..n, 1..m};           # Loss matrix for player 2

var p{1..n};                   # Probability player 1 selects strategy i
var q{1..m};                   # Probability player 2 selects strategy j

subject to
    opt1 {i in 1..n}:           # Optimality conditions for player 1
        0 <= p[i] complements sum{j in 1..m} A[i,j] * q[j] >= 1;

    opt2 {j in 1..m}:           # Optimality conditions for player 2
        0 <= q[j] complements sum{i in 1..n} B[i,j] * p[i] >= 1;
Part V

Numerical Optimization IV: Extensions
Global Optimization

I need to find the GLOBAL minimum!

- use any NLP solver (often work well!)
- use the multi-start trick from previous slides
- global optimization based on branch-and-reduce: BARON
  - constructs global underestimators
  - refines region by branching
  - tightens bounds by solving LPs
  - solve problems with 100s of variables
- “voodoo” solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic
Derivative-Free Optimization

My model does not have derivatives!

- Change your model ... good models have derivatives!
- pattern-search methods for \( \min f(x) \)
  - evaluate \( f(x) \) at stencil \( x_k + \Delta M \)
  - move to new best point
  - extend to NLP; some convergence theory h
- matlab: NOMADm.m; parallel APPSPACK

- solvers based on building interpolating quadratic models
  - DFO project on www.coin-or.org
  - Mike Powell’s NEWUOA quadratic model

- “voodoo” solvers: genetic algorithm & simulated annealing
  no convergence theory ... usually worse than deterministic
Optimization with Integer Variables

Mixed-Integer Nonlinear Program (MINLP)

- modeling discrete choices $\Rightarrow$ 0–1 variables
- modeling integer decisions $\Rightarrow$ integer variables
  e.g. number of different stocks in portfolio (8-10)
  not number of beers sold at Goose Island (millions)

MINLP solvers:

- branch (separate $z_i = 0$ and $z_i = 1$) and cut
- solve millions of NLP relaxations: MINLPBB, SBB
- outer approximation: iterate MILP and NLP solvers
  BONMIN (COIN-OR) & FilMINT on NEOS
Portfolio Management

- \( N \): Universe of asset to purchase
- \( x_i \): Amount of asset \( i \) to hold
- \( B \): Budget

\[
\text{minimize } u(x) \quad \text{subject to } \sum_{i \in N} x_i = B, \quad x \geq 0
\]
Portfolio Management

- $N$: Universe of asset to purchase
- $x_i$: Amount of asset $i$ to hold
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\[
\text{minimize } u(x) \quad \text{subject to } \sum_{i \in N} x_i = B, \quad x \geq 0
\]

- **Markowitz**: $u(x) \overset{\text{def}}{=} -\alpha^T x + \lambda x^T Q x$
  - $\alpha$: maximize expected returns
  - $Q$: variance-covariance matrix of expected returns
  - $\lambda$: minimize risk; aversion parameter
More Realistic Models

- $b \in \mathbb{R}^{|N|}$ of “benchmark” holdings
- Benchmark Tracking: $u(x) \overset{\text{def}}{=} (x - b)^T Q (x - b)$
  - Constraint on $\mathbb{E}[\text{Return}]: \alpha^T x \geq r$
More Realistic Models

- $b \in \mathbb{R}^{|N|}$ of “benchmark” holdings

- **Benchmark Tracking:** $u(x) \overset{\text{def}}{=} (x - b)^T Q(x - b)$
  - Constraint on $\mathbb{E}[\text{Return}]: \alpha^T x \geq r$

- **Limit Names:** $|i \in N : x_i > 0| \leq K$
  - Use binary indicator variables to model the implication $x_i > 0 \Rightarrow y_i = 1$
  - Implication modeled with variable upper bounds:

  $$x_i \leq By_i \quad \forall i \in N$$

- $\sum_{i \in N} y_i \leq K$
Optimization Conclusions

Optimization is General Modeling Paradigm
- linear, nonlinear, equations, inequalities
- integer variables, equilibrium, control

AMPL (GAMS) Modeling and Programming Languages
- express optimization problems
- use automatic differentiation
- easy access to state-of-the-art solvers

Optimization Software
- open-source: COIN-OR, IPOPT, Soplex, & ASTROS (soon)
- current solver limitations on laptop:
  - 1,000,000 variables/constraints for LPs
  - 100,000 variables/constraints for NLPs/NCPs
  - 100 variables/constraints for global optimization
  - 500,000,000 variable LP on BlueGene/P