Initiative for Computational Economics
Numerical Methods for Solving Auctions II

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20 July 2012

Acknowledgements

This presentation builds on published and ongoing work with **Timothy P. Hubbard** and draws on research we have completed with **René Kirkegaard** and we are continuing with **Ken Judd**.

Remember the Asymmetric First-Price Auction

Bidder n maximizes

$$\mathbb{E}(U_n|s_n) = (V_n - s_n) \prod_{m \neq n} F_m[\varphi_m(s_n)]$$

which, after playing with the FOCs, led to

$$\frac{1}{\varphi_n(s) - s} = \sum_{m \neq n} \frac{f_m[\varphi_m(s)]}{F_m[\varphi_m(s)]} \varphi'_m(s),$$

or, equivalently (after some algebra)

$$\varphi'_n(s) = \frac{F_n[\varphi_n(s)]}{f_n[\varphi_n(s)]} \left\{ \left[\frac{1}{(N-1)} \sum_{m=1}^{N} \frac{1}{\varphi_m(s) - s} \right] - \frac{1}{\varphi_n(s) - s} \right\}.$$

Two Sets of Boundary Conditions Also Exist

Left-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\underline{v}) = \underline{v}$ for all $n = 1, 2, \ldots, N$.

Right-Boundary Condition on Inverse-Bid Functions: $\varphi_n(\bar{s}) = \bar{v}$ for all $n = 1, 2, \ldots, N$.

Why Is This Interesting?

- a system of nonlinear differential equations obtain;
- no longer an initial value problem (as in symmetric case), but rather a two-point boundary value problem;
- $\bar{s}$ is unkown *a priori* and determines domain of solutions;
- boundary value problem is *overidentified*;
- we know some characteristics that the solutions must respect, for example, rationality and monotonicity;
- Lipschitz condition does not hold at $\underline{v}$!

Consider Shooting Methods
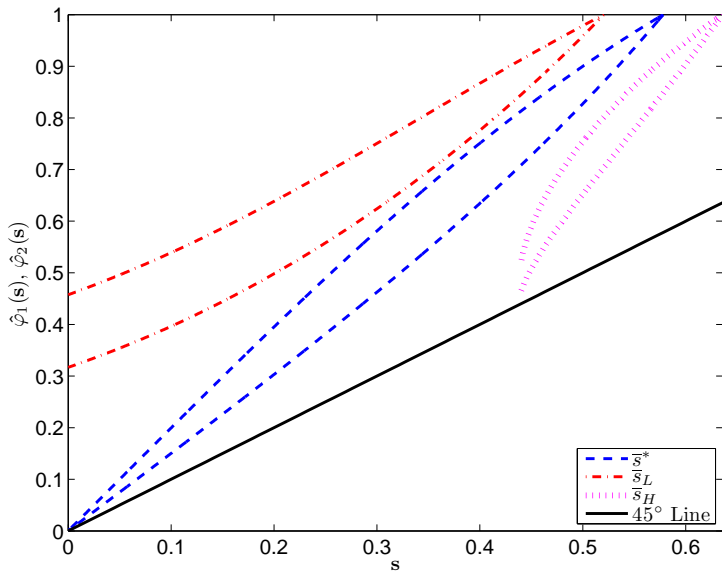
Coalition of 3 versus Coalition of 2

We considered the MMRS (1994) example of coalitions.
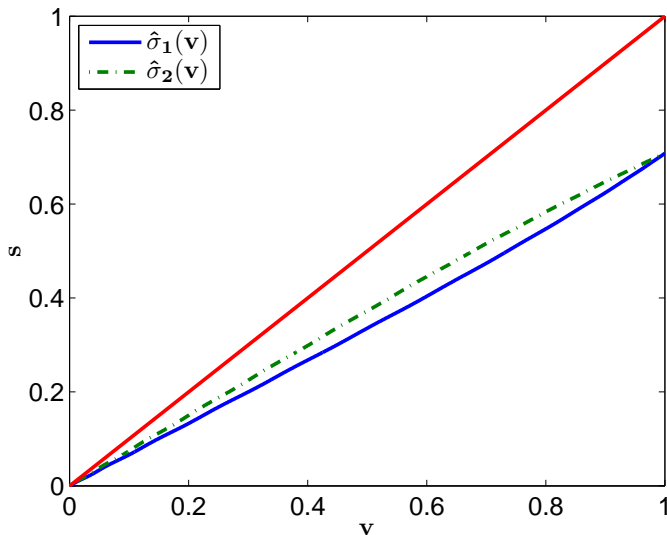
Raining on the Shooting Parade

I was very careful about the example I chose—with uniform $F_0(\cdot)$ the maximum valuations from each coalition imply asymmetric power distributions, one of the only cases with closed-form solutions.

Nearly all researchers who used shooting methods noted that the algorithm was very sensitive and instable.

Recently, Fibich and Gavish (2011, GEB) have proven analytically that the inherent instability is not a "technical issue," but rather an analytic property of backward integration in this setting.

Furthermore, shooting methods are very costly (in terms of time), require more advanced programming techniques, and typically involve a lot of "fiddling."

Projection Methods: An Alternative

A projection method is a general strategy to approximating a true, but unknown, function by a finite number of approximating functions.

Idea: the true solution is approximated by a finite combination of simple, known functions.

In our setting, this means researchers would first choose a basis to approximate the solutions to each inverse-bid function.

Basis-Related Choices

The full basis for the space of candidate solutions should be rich (flexible) enough to approximate any function relevant to the problem, which will be represented and approximated as a linear combination of basis functions.

The researcher would then fix the flexibility of the approximation by deciding how many basis elements to include: in short, the researcher must select the degree of the approximation.

Fixing the degree transforms an infinite-dimensional problem into a finite-dimensional one, where only the coefficients of the basis functions need then to be found; if the basis is a good choice, then larger degrees should yield better approximations.

Other Choices: Residual Function and Norm

The researcher must also decide on an appropriate residual function to evaluate how closely the approximation represents the true solution.

The goal of projection methods is to find a set of coefficients which make some norm of the residual function as close to zero as possible.

After solving, the researcher can then verify the quality of the candidate solution and choose either to increase the degree of the approximation or, if that fails, then to use a different basis.

Spectral Methods

Spectral methods use bases where each element is nonzero almost everywhere, as with trigonometric bases and orthogonal polynomials.

In the case of an asymmetric first-price auction problem, consider approximating each inverse-bid function by a truncated series expansion

$$\hat{\varphi}_n(s) = \sum_{k=0}^{K} \alpha_{n,k} \mathbb{P}_k(s), \ s \in [\bar{v}, \bar{s}], \ n = 1, 2, \dots, N$$

where $\mathbb{P}_k(s)$ is some basis functions (which are typically chosen to be polynomials) and the $\alpha_{n,k}$s are referred to as the *spectral coefficients*.

Least-Squares Problem

For economists, perhaps the most intuitive spectral method is that of least-squares. It is compelling to us: we have reduced the problem of solving a functional equation to solving a nonlinear minimization problem, a problem with which we have considerable experience.

Consider selecting a large number T of grid points from the interval $[\underline{v}, \bar{s}]$.

Remember the Asymmetric IPVP Problem

Define

$$G_n(s; \bar{s}, \boldsymbol{\alpha}) \equiv 1 - [\hat{\varphi}_n(s) - s] \sum_{m \neq n} \frac{f_m[\hat{\varphi}_m(s)]}{F_m[\hat{\varphi}_m(s)]} \hat{\varphi}'_m(s)$$

where $\boldsymbol{\alpha}$ denotes a vector that collects the $N \times (K + 1)$ coefficients of the polynomials.

Note that once the basis function has been chosen, $\hat{\varphi}'_m(\cdot)$ are implied by the choice: literally, just take the derivative of each basis element.

Ideally . . .

In an exact solution, $G_n(s; \bar{s}, \boldsymbol{\alpha})$ should equal zero for all bidders and at any bid $s \in [\underline{v}, \bar{s}]$ and our boundary constraints

$$\varphi_n(\underline{v}) = \underline{v}$$

and

$$\varphi_n(\bar{s}) = \bar{v}$$

will be satisfied for all $n = 1, \ldots, N$.

Bajari (2001, ET)

The problem is to estimate $\bar{s}$ as well as the $\alpha_{n,k}$s for all $n = 1, 2, \ldots, N$ and $k = 0, 1, \ldots, K$.

The system can be evaluated at each grid point and the parameters can be chosen to minimize the following criterion function:

$$H(\bar{s}, \boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \boldsymbol{\alpha})]^2 + \sum_{n=1}^{N} \left[\hat{\varphi}_n(\underline{v}) - \underline{v}\right]^2 + \sum_{n=1}^{N} [\hat{\varphi}_n(\bar{s}) - \bar{v}]^2$$

Bajari (2001, ET)

In practice, Bajari chose

- $K = 5$
- uniformly-spaced grid
- ordinary polynomials
- used a nonlinear least-squares algorithm to select $\bar{s}$ and $\boldsymbol{\alpha}$ by minimizing a modified version of the previous objective

$$\tilde{H}(\bar{s}, \boldsymbol{\alpha}) \equiv \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \boldsymbol{\alpha})]^2 + T \sum_{n=1}^{N} \left[\hat{\varphi}_n(\underline{v}) - \underline{v}\right]^2 + T \sum_{n=1}^{N} [\hat{\varphi}_n(\bar{s}) - \bar{v}]^2$$

which adds weight to the boundary conditions.

Hubbard and Paarsch (2009, IJIO)

Modified the approach we just discussed by

- using Chebyshev points;
- using Chebyshev basis;
- imposing boundary conditions as constraints;
- imposing shape constraints.

Thus, the problem becomes . . .

Hubbard and Paarsch (2009, IJIO) Continued . . .

$$\min_{\{\bar{s},\boldsymbol{\alpha}\}} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \boldsymbol{\alpha})]^2$$

subject to each of these, for all $n = 1, 2, \ldots, N$,

1. $\underline{v} \leq \bar{s} \leq \bar{v}$
2. $\varphi_n(\underline{v}) = \underline{v}$
3. $\varphi_n(\bar{s}) = \bar{v}$
4. $\varphi_n(s_{j-1}) \leq \varphi_n(s_j)$ for uniform grid $j = 2, \ldots, J$.

Hubbard, Kirkegaard, and Paarsch (2012, CE)

HKP took this improved approach and pushed it farther by levering other information concerning the critical boundary points.

Fibich, Gavious, and Sela (2002, Econ Let) proved the following properties concerning the high and low types, the first of which follows directly from the first-order conditions:

1. $(\bar{v} - \bar{s}) \sum_{m \neq n} f_m(\bar{v}) \varphi'_m(\bar{s}) = 1$ for all $n = 1, 2, \ldots, N$.
2. If $f_n(\underline{v}) \in \mathbb{R}_{++}$ and $\varphi_n(s)$ is differentiable at $s = \underline{v}$ for all $n = 1, 2, \ldots, N$, then $\varphi'_n(\underline{v}) = [N/(N-1)]$.

HKP took the HP (2009, IJIO) approach and imposes these as well to get . . .

Hubbard, Kirkegaard, and Paarsch (2012, CE)

$$\min_{\{\bar{s}, \boldsymbol{\alpha}\}} \sum_{n=1}^{N} \sum_{t=1}^{T} [G_n(s_t; \bar{s}, \boldsymbol{\alpha})]^2$$

subject to each of these, for all $n = 1, 2, \ldots, N$,

1. $\underline{v} \leq \bar{s} \leq \bar{v}$
2. $\varphi_n(\underline{v}) = \underline{v}$
3. $\varphi_n(\bar{s}) = \bar{v}$
4. $\sum_{m \neq n} (\bar{v} - \bar{s}) f_m(\bar{v}) \varphi'_m(\bar{s}) = 1$
5. $\varphi'_n(\underline{v}) = [N/(N-1)]$
6. $\varphi_n(s_{j-1}) \leq \varphi_n(s_j)$ for uniform grid $j = 2, \ldots, J$.

HKP (2012, CE): Side Note on Collocation

Under this approach, there are $4N$ conditions (constraints) in total and TN points that enter the objective function.

By comparison, there are $N(K + 1) + 1$ parameters to be estimated—the parameters in α plus $\bar{s}$.

For the number of conditions (boundary and first-order together) to equal the number of unknowns

$$N(T + 4) = N(K + 1) + 1$$

or

$$(T + 4) = (K + 1) + \frac{1}{N}.$$

Because N weakly exceeds two at auctions, and T and K are integers, this equality cannot hold for any (T, K) choice.

HKP (2012, CE): Initial Guess

However, when comparing the $N(K+1)+1$ parameters with the $4N$ conditions, note that, if K equals three and all the conditions are satisfied, then only one degree of freedom remains.

One criticism of the polynomial approach (and projection-based methods in general) is that it works well, if the practitioner has a good initial guess.

When K equals three, the researcher obtains an initial guess that already satisfies some theoretical properties at essentially no cost because there is only one free parameter, $\bar{s}$, to minimize the nonlinear least-squares objective.

Hubbard and Paarsch (2009, IJIO)

HP (2009, IJIO) have provided a nice example of projection methods as, even if shooting methods could work reliably, they would not work on the HP problem.

This example involve bid preference programs which is now a well studied topic, especially among structural econometricians.

- Marion (2007, JPubE)
- Krasnokutskaya and Seim (2011, AER)

In these programs, bids of preferred firms are typically scaled by some discount factor which is one plus a preference rate denoted ρ. Suppose there are N_1 preferred bidders and N_2 typical (nonpreferred) bidders, where $(N_1 + N_2)$ equals N.

HP (2009, IJIO) Example

Under this program, probability of winning for a class 1 bidder,

$$\Pr(\mathsf{win}|b_1) = (1 - F_1[\varphi_1(b_1)])^{N_1-1} \left(1 - F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\right)^{N_2},$$

while for a class 2 bidder probability of winning is

$$\Pr(\mathsf{win}|b_2) = [1 - F_1(\varphi_1[(1+\rho)b_2])]^{N_1}(1 - F_2[\varphi_2(b_2)])^{N_2-1}.$$

HP Example Continued . . .

Each firm then chooses its bid b to maximize its

$$\mathbb{E}(U_i|b_i) = (b_i - C_i)\Pr(\text{win}|b_i)$$

which yields two FOCs

$$\frac{\partial \mathbb{E}(U_1|b_1)}{\partial b_1} = 1 - [b_1 - \varphi_1(b_1)]\left[\frac{(N_1 - 1)f_1\left[\varphi_1(b_1)\right]\varphi_1'(b_1)}{1 - F_1\left[\varphi_1(b_1)\right]} + \frac{N_2 f_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]\frac{1}{1+\rho}\varphi_2'\left(\frac{b_1}{1+\rho}\right)}{1 - F_2\left[\varphi_2\left(\frac{b_1}{1+\rho}\right)\right]}\right] = 0$$

and

$$\frac{\partial \mathbb{E}(U_2|b_2)}{\partial b_2} = 1 - [b_2 - \varphi_2(b_2)]\left[\frac{N_1 f_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)(1+\rho)\varphi_1'\left[(1+\rho)b_2\right]}{1 - F_1\left(\varphi_1\left[(1+\rho)b_2\right]\right)} + \frac{(N_2 - 1)f_2\left[\varphi_2(b_2)\right]\varphi_2'(b_2)}{1 - F_2\left[\varphi_2(b_2)\right]}\right] = 0.$$

HP Example Continued . . .

**Right-Boundary Conditions (on Inverse-Bid Functions):**

- for all nonpreferred bidders of class 2, $\varphi_2(\bar{c}) = \bar{c}$;
- for all preferred bidders of class 1, $\varphi_1(\bar{b}) = \bar{c}$, where $\bar{b} = \bar{c}$ if $N_1 > 1$, but when $N_1 = 1$, then $\bar{b}$ is determined by

$$\bar{b} = \underset{b}{\operatorname{argmax}} \left[(b - \bar{c}) \left(1 - F_2 \left[\varphi_2 \left(\frac{b}{1 + \rho} \right) \right] \right)^{N_2} \right].$$
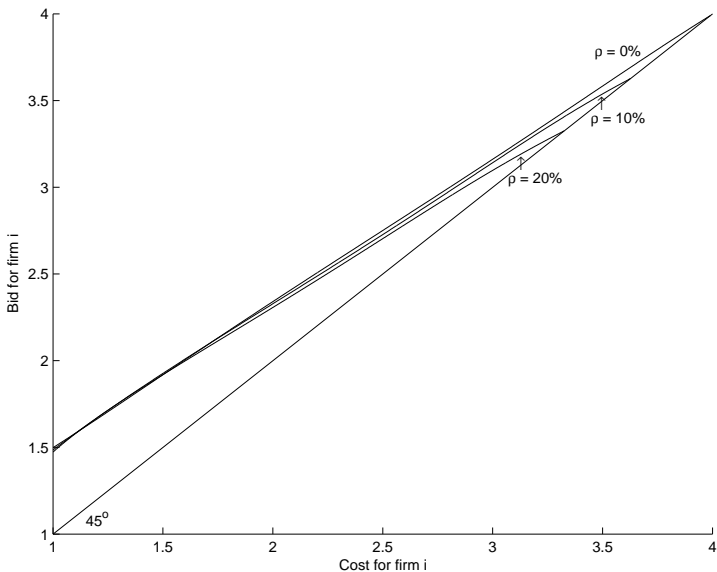
Left-Boundary Conditions (on Inverse-Bid Functions): there exists an unknown bid $\underline{b}$ such that

- for all nonpreferred bidders of class 2, $\varphi_2(\underline{b}) = \underline{c}$;
- for all preferred bidders of class 1, $\varphi_1\left[(1 + \rho)\underline{b}\right] = \underline{c}$.

Endogenous Changes in Preferred Behavior
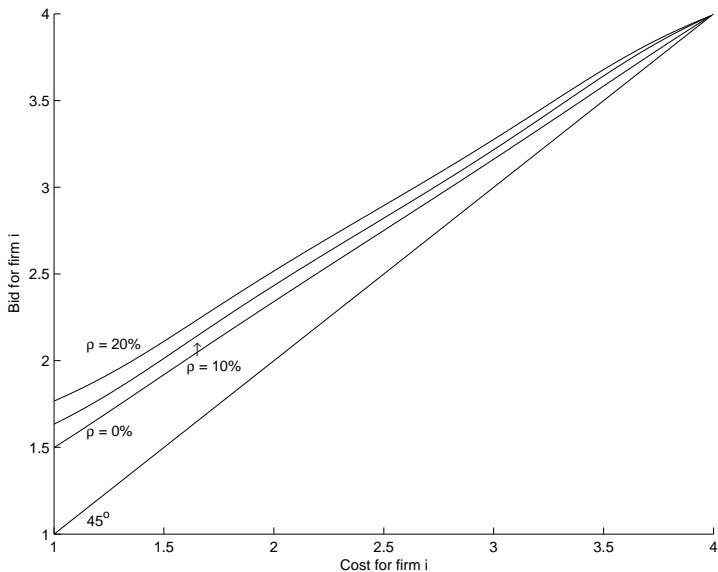
Endogenous Changes in Nonpreferred Behavior

Are We Comfortable with Less Control?

While shooting methods have so many issues, a distinct advantage is that the error tolerance (at $\underline{v}$—the point we were shooting to) can be controlled explicitly. The researcher must specify this beforehand.

Of course, whether this can be achieved and whether the shooting approach will be successful is another issue.

Nonetheless, that control is a nice feature. How does one know whether a polynomial approximation is sufficient? Should one select a higher degree polynomial?

Regardless of the Approach, Theory Tells Us More

I mentioned earlier that we had more information concerning the derivatives of the bid functions at the boundaries.

We can also use theory to inform us about some qualitative properties of the bid functions. In the projection approach, this can also help inform us about whether we are capturing essential features of the solution.

This comes from Hubbard, Kirkegaard, and Paarsch (2012, CE)—HKP.

Comparing Bidder Behavior

Define

$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s).$$

When $K = 3$, this becomes

$$D_{n,m}(s) = \hat{\varphi}_n(s) - \hat{\varphi}_m(s) = \left[\frac{f_n(\bar{v}) - f_m(\bar{v})}{f_n(\bar{v}) f_m(\bar{v})} \right] \left[\frac{s^2(\bar{s} - s)}{\bar{s}^2(\bar{v} - \bar{s})} \right].$$

Proposition:
Assume (i) $f_n(\underline{v}) \in \mathbb{R}_{++}$ and (ii) $\varphi_n(s)$ is a polynomial of degree K, $K \geq 3$, with real coefficients that satisfy Conditions 1a, 1b, 2a, and 2b, for all $n = 1, 2, \ldots, N$. If $f_n(\bar{v}) \neq f_m(\bar{v})$, then $\varphi_n(s)$ and $\varphi_m(s)$ cross at most $(K - 3)$ times on $(\underline{v}, \bar{s})$, $m, n = 1, 2, \ldots, N$.

Example 2
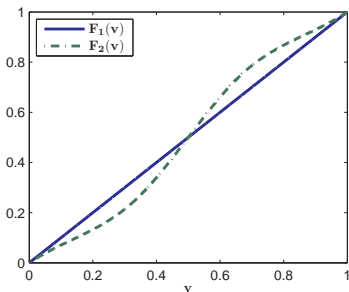
(a) CDFs
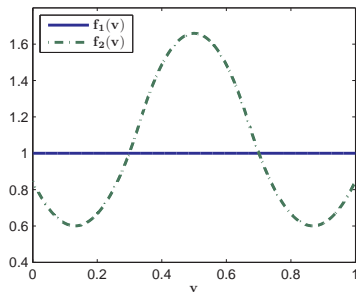
(b) PDFs

Example 2: $\hat{D}_{1,2}(s)$ for Various K
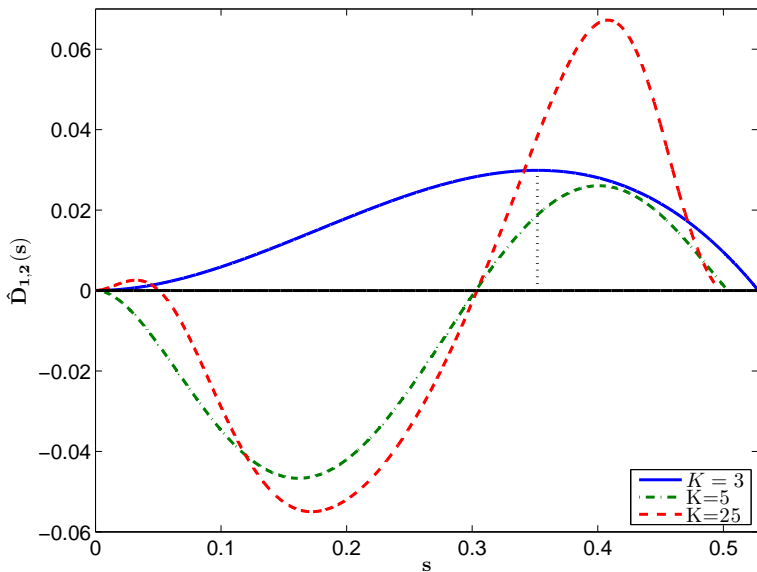
Exogenous and Endogenous Ratios

Let

$$P_{n,m}(v) = \frac{F_m(v)}{F_n(v)}, \; v \in (\underline{v}, \bar{v}]$$

measure bidder n's strength relative to bidder m at a given v.

Let

$$R_{n,m}(v) = \frac{U_n(v)}{U_m(v)}, \; v \in [\underline{v}, \bar{v}]$$

denote bidder n's equilibrium pay-off relative to bidder j's equilibrium pay-off at a given v; i.e.,

$$U_n(v) = (v_n - s) \prod_{m \neq n} F_m \left[\varphi_m(s) \right].$$

Properties of the Ratios

- Ratios versus Bids

$$R_{n,m}(v) \gtreqqless P_{n,m}(v) \iff \sigma_n(v) \gtreqqless \sigma_m(v), \text{ for } v \in (\underline{v}, \bar{v}].$$

- Endogenous versus Exogenous Ratios

$$R'_{n,m}(v) \gtreqqless 0 \iff R_{n,m}(v) \gtreqqless P_{n,m}(v), \text{ for } v \in (\underline{v}, \bar{v}].$$

- Right-Boundary Condition

$$R_{n,m}(\bar{v}) = P_{n,m}(\bar{v}) = 1.$$

- Left-Boundary Condition

$$\lim_{v \to \underline{v}} R_{n,m}(v) = \frac{f_m(\underline{v})}{f_n(\underline{v})} = \lim_{v \to \underline{v}} P_{n,m}(v).$$

Implications for Approximations

1. Slope:
   If $P_{n,m}(v) = \hat{R}_{n,m}(v)$ (i.e., $\hat{\sigma}_n(v) = \hat{\sigma}_m(v)$), $\hat{R}_{n,m}$ should be *flat*. This is true anytime bids coincide (for any $v > \underline{v}$, including $\bar{v}$).

2. Location:
   $P_{n,m}(v) = R_{n,m}(v)$ *at most* once between any two peaks of $P_{n,m}$. With diminishing wave property they *must* cross between any two peaks (not counting v equals $\bar{v}$).

Implications for Approximations
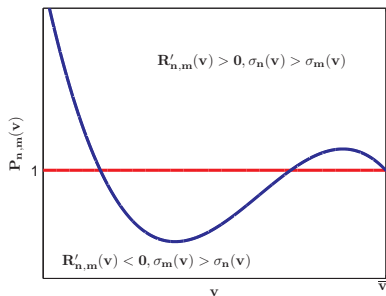
(c) Exogenous Ratio
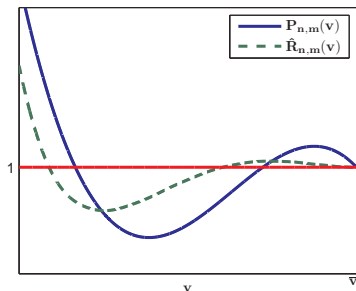
(d) Both Ratios

Example 2 Continued . . .
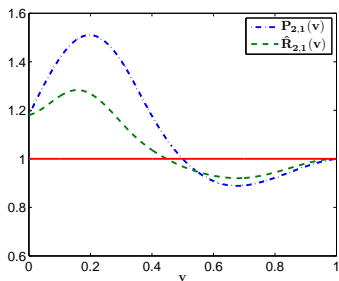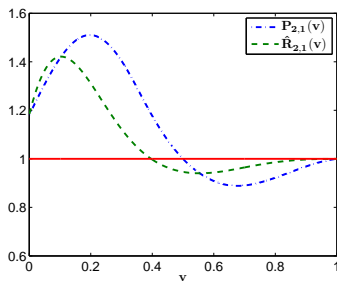
(e) $K = 5$ (f) $K = 25$

Example 3: Diminishing Wave Property Satisfied
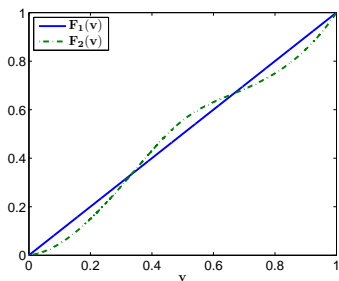
(g) CDFs
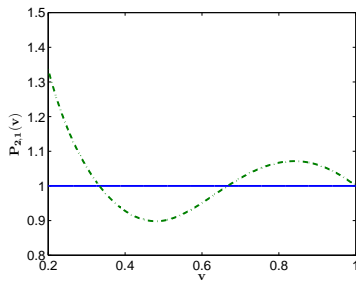
(h) Exogenous Ratio

Example 3 Approximations
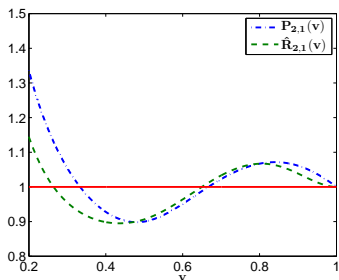
(i) $K = 5$
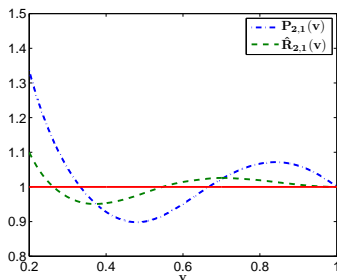
(j) $K = 25$

Are Good Solutions Really that Important?

I hope that you see that these graphical "tests" can be used to evaluate some qualitative features of the approximation.

But, really, how important are these small differences in the approximated (inverse) bid functions?

Well, let us simulate some auctions and find out . . .

Results of a Simulation Study: One Trillion Auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$\mathbb{E}(U_1)$	$\mathbb{E}(U_2)$
	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K = 4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K = 5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K = 10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
Ex. 2	$K = 15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K = 20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
	$K = 3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K = 4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K = 5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K = 10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
Ex. 3	$K = 15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K = 20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

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	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	$K = 30$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
	SPA	0.3399	0.0000	0.4867	0.5133	0.1601	0.1733

Poor approximation $\Rightarrow$ wrong revenue ranking.

Results of a Simulation Study: One Trillion Auctions

	Order of $\hat{\varphi}_n$	Expected Revenue	Proportion of Inefficiencies	Prop. Wins Player 1	Prop. Wins Player 2	$E(U_1)$	$E(U_2)$
	$K = 3$	0.3563	0.0193	0.4807	0.5193	0.1553	0.1436
	$K = 4$	0.3473	0.0223	0.5209	0.4791	0.1645	0.1434
	$K = 5$	0.3458	0.0227	0.5089	0.4911	0.1634	0.1459
	$K = 10$	0.3431	0.0337	0.5028	0.4972	0.1635	0.1481
Ex. 2	$K = 15$	0.3432	0.0338	0.5027	0.4973	0.1634	0.1481
	$K = 20$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 25$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	$K = 30$	0.3432	0.0338	0.5026	0.4974	0.1634	0.1481
	SPA	0.3445	0.0000	0.5000	0.5000	0.1555	0.1555
	$K = 3$	0.3475	0.0721	0.5491	0.4509	0.1568	0.1568
	$K = 4$	0.3452	0.0670	0.5287	0.4713	0.1584	0.1613
	$K = 5$	0.3364	0.0557	0.5041	0.4959	0.1605	0.1678
	$K = 10$	0.3334	0.0704	0.4964	0.5036	0.1632	0.1699
Ex. 3	$K = 15$	0.3330	0.0719	0.4947	0.5053	0.1634	0.1702
	$K = 20$	0.3328	0.0725	0.4944	0.5056	0.1635	0.1703
	$K = 25$	0.3328	0.0726	0.4943	0.5057	0.1635	0.1703
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$$\mathbb{E}(U_n^{\mathrm{SP}}) - \mathbb{E}(U_m^{\mathrm{SP}}) = \mu_n - \mu_m, \text{ but not in FPAs here.}$$

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SPA $\Rightarrow$ bigger pie.

In these, for seller, SPA > FPA, but for bidders, FPA > SPA
(collectively, but not individually)

Results of a Simulation Study: One Trillion Auctions

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SPA $\rightarrow$ FPA $\Rightarrow$ small change in probability of winning

Player 1 wins $\approx$1% more of the time than player 2 in FPA

Results of a Simulation Study: One Trillion Auctions

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SPA $\rightarrow$ FPA $\Rightarrow$ bigger change payoff;
$\mathbb{E}(U_1) > \mathbb{E}(U_2)$ by $\approx$10% in FPA.

It's a Small World

I think auctions are an area where considerable interplay between numerical, applied, and theoretical work can exist.

Theory informs numerical work.

Numerical work motivates new theory.

Empirical work relies on theory and numerical work.

Theory and numerical work are guided by new empirical observations.

APVP example

Consider a first-price auction with no reserve price involving two bidders.

The bidders draw valuations from a joint distribution $F_{\mathbf{V}}(v_1, v_2)$ which has compact support $[\underline{v}, \bar{v}]^2$. We employ Sklar's Theorem which states that a copula $C[F_1(v_1), F_2(v_2)]$ always exists, and is a unique function linking $F_{\mathbf{V}}(v_1, v_2)$ with $F_1(v_1)$ and $F_2(v_2)$.

We assume a Frank copula with dependence parameter θ set such that Kendall's τ equals 0.5 which implies non-negligible statistical dependence between V_1 and V_2.

Under these conditions, we have a model in the APVP.

Example 4

Assume further that valuations for bidder 1 have a uniform marginal distribution
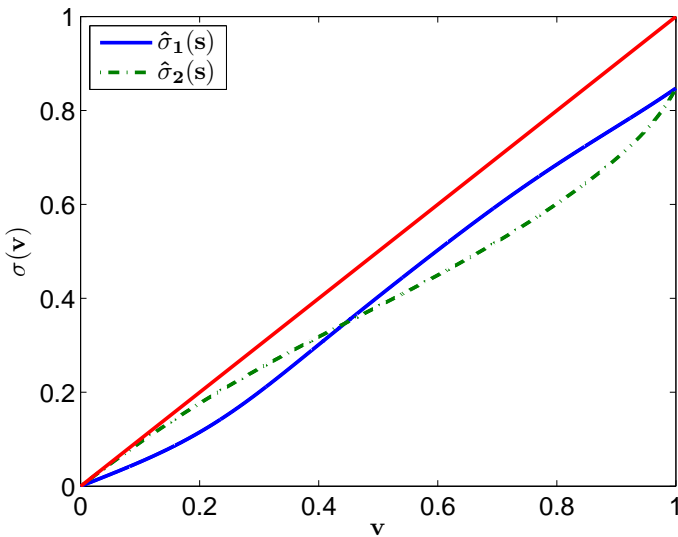
$$F_1(v) \sim F(v; 1, 1)$$

while valuations from bidder 2 are distributed via the following beta-uniform mixture marginal distribution:

$$F_2(v) \sim \gamma F(v; 1, 1) + (1 - \gamma)F(v; 2, 2)$$

with the weight γ equal to 0.1.

Note that $F_2(v)$ first-order stochastically dominates $F_1(v) \Rightarrow$ in IPVP, weakness leads aggression holds.

Example 4: Asymmetry and Affiliation

Things to Take from Our Discussions

When bidders are *ex ante* heterogenous at first-price auctions, closed-form solutions often do not exist and there is a role for numerical methods.

We discussed two of them extensively: backwards shooting methods and perturbation methods.

Theory can be used to provide some validation of the solution—regardless of the approach taken.

There is room for improvements: I shall circulate a preliminary version of a paper I am working on with Timothy P. Hubbard in which we try to bring everything in the literature together and discuss some future directions for research.