Perturbation of Growth Model

```
Off[General::"spell1"]
Off[General::"spell"]
SessionTime[]
```

0.2503600

Deterministic Model

▪ setup

```
u[x_] = Log[x];
f[x_] = x + A x^α;
α = .25; β = .95;
```

```
f'[1]
```

$1 + 0.25\,A$

```
Solve[% == 1 / β, A]
```

{{A → 0.210526}}

```
A = A //. %[[1]]
```

0.210526

```
kplus = f[k] - c[k]
cplus = c[kplus]
fp[x_] = D[f[x], x]
EulerEq = u'[c[k]] - β u'[cplus] fp[kplus]
```

$0.210526\,k^{0.25} + k - c[k]$

$c[0.210526\,k^{0.25} + k - c[k]]$

$1 + \frac{0.0526316}{x^{0.75}}$

$$\frac{1}{c[k]} - \frac{0.95\left(1 + \frac{0.0526316}{(0.210526\,k^{0.25}+k-c[k])^{0.75}}\right)}{c[0.210526\,k^{0.25} + k - c[k]]}$$

```
ss = {k → 1, 1. → 1, 0. → 0}
```

{k → 1, 1. → 1, 0. → 0}

```
EulerEq //. ss
```

$$\frac{1}{c[1]} - \frac{0.95\left(1 + \frac{0.0526316}{(1.21053 - c[1])^{0.75}}\right)}{c[1.21053 - c[1]]}$$

```
c[1] = css = f[1] - 1
```

0.210526

```
EulerEq //. ss // Simplify
```

8.88178×10^{-16}

```
sol = {};
```

■ k pert

```
D[EulerEq, k]
```

$$\frac{0.0375\left(1 + \frac{0.0526316}{k^{0.75}} - c'[k]\right)}{(0.210526\,k^{0.25} + k - c[k])^{1.75}\, c[0.210526\,k^{0.25} + k - c[k]]} - \frac{c'[k]}{c[k]^2} + \frac{0.95\left(1 + \frac{0.0526316}{(0.210526\,k^{0.25} + k - c[k])^{0.75}}\right)\left(1 + \frac{0.0526316}{k^{0.75}} - c'[k]\right) c'[0.210526\,k^{0.25} + k - c[k]]}{c[0.210526\,k^{0.25} + k - c[k]]^2}$$

```
% //. ss
```

$0.178125\,(1.05263 - c'[1]) - 22.5625\,c'[1] + 22.5625\,(1.05263 - c'[1])\,c'[1]$

```
% // Simplify
```

$-22.5625\,(-0.116233 + c'[1])\,(0.0714964 + c'[1])$

```
Solve[% == 0, c'[1]]
```

{{c′[1] → −0.0714964}, {c′[1] → 0.116233}}

```
sol = Union[sol, %[[2]]]
```

{c′[1] → 0.116233}

■ k pert - degree 2

```
D[EulerEq, k, k]
```

$$-\frac{0.065625\left(1+\frac{0.0526316}{k^{0.75}}-c'[k]\right)^2}{(0.210526\,k^{0.25}+k-c[k])^{2.75}\,c[0.210526\,k^{0.25}+k-c[k]]}+$$
$$\frac{2\,c'[k]^2}{c[k]^3}-\frac{0.075\left(1+\frac{0.0526316}{k^{0.75}}-c'[k]\right)^2 c'[0.210526\,k^{0.25}+k-c[k]]}{(0.210526\,k^{0.25}+k-c[k])^{1.75}\,c[0.210526\,k^{0.25}+k-c[k]]^2}-$$
$$\frac{1.9\left(1+\frac{0.0526316}{(0.210526\,k^{0.25}+k-c[k])^{0.75}}\right)\left(1+\frac{0.0526316}{k^{0.75}}-c'[k]\right)^2 c'[0.210526\,k^{0.25}+k-c[k]]^2}{c[0.210526\,k^{0.25}+k-c[k]]^3}+$$
$$\frac{0.0375\left(-\frac{0.0394737}{k^{1.75}}-c''[k]\right)}{(0.210526\,k^{0.25}+k-c[k])^{1.75}\,c[0.210526\,k^{0.25}+k-c[k]]}+$$
$$\frac{0.95\left(1+\frac{0.0526316}{(0.210526\,k^{0.25}+k-c[k])^{0.75}}\right)c'[0.210526\,k^{0.25}+k-c[k]]\left(-\frac{0.0394737}{k^{1.75}}-c''[k]\right)}{c[0.210526\,k^{0.25}+k-c[k]]^2}-\frac{c''[k]}{c[k]^2}+$$
$$\frac{0.95\left(1+\frac{0.0526316}{(0.210526\,k^{0.25}+k-c[k])^{0.75}}\right)\left(1+\frac{0.0526316}{k^{0.75}}-c'[k]\right)^2 c''[0.210526\,k^{0.25}+k-c[k]]}{c[0.210526\,k^{0.25}+k-c[k]]^2}$$

```
% //. ss
```

$$-0.311719\,(1.05263-c'[1])^2-1.69219\,(1.05263-c'[1])^2\,c'[1]+$$
$$214.344\,c'[1]^2-214.344\,(1.05263-c'[1])^2\,c'[1]^2+0.178125\,(-0.0394737-c''[1])+$$
$$22.5625\,c'[1]\,(-0.0394737-c''[1])-22.5625\,c''[1]+22.5625\,(1.05263-c'[1])^2\,c''[1]$$

```
% //. sol
```

$$-0.0891494+2.80064\,(-0.0394737-c''[1])-2.77875\,c''[1]$$

```
% // Simplify
```

$$-0.199701-5.57939\,c''[1]$$

```
Solve[% == 0, c''[1]]
```

$$\{\{c''[1]\to -0.0357926\}\}$$

```
sol = Union[sol, %[[1]]] // Simplify
```

$$\{c'[1]\to 0.116233,\ c''[1]\to -0.0357926\}$$

■ k pert - degree 3

```
D[EulerEq, k, k, k] //. ss //. sol // Simplify
```

$$0.370006-6.83767\,c^{(3)}[1]$$

```
Solve[% == 0, c'''[1]];
```

```
sol = Union[sol, %[[1]]] // Simplify;
```

```
sol
```

$\{c'[1] \to 0.116233,\ c''[1] \to -0.0357926,\ c^{(3)}[1] \to 0.0541129\}$

```
Series[c[x], {x, 1, 3}] //. sol
```

$0.210526 + 0.116233\ (x-1) - 0.0178963\ (x-1)^2 + 0.00901882\ (x-1)^3 + O[x-1]^4$

■ k pert - more

```
Do[
 dκ = D[EulerEq, {k, i}] //. ss //. sol // Simplify;
 solκ = Solve[dκ == 0, Derivative[i][c][1]];
  sol = Union[sol, solκ[[1]]] // Simplify,
 {i, 4, 10}]
```

```
sol
```

$\{c'[1] \to 0.116233,\ c''[1] \to -0.0357926,\ c^{(3)}[1] \to 0.0541129,$
$c^{(4)}[1] \to -0.135759,\ c^{(5)}[1] \to 0.47631,\ c^{(6)}[1] \to -2.14781,$
$c^{(7)}[1] \to 11.8364,\ c^{(8)}[1] \to -77.0954,\ c^{(9)}[1] \to 579.493,\ c^{(10)}[1] \to -4937.49\}$

```
Do[Print[SessionTime[]];
  dκ = D[EulerEq, {k, i}] //. ss //. sol // Simplify;
  solκ = Solve[dκ == 0, Derivative[i][c][1]];
   sol = Union[sol, solκ[[1]]] // Simplify,
  {i, 11, 15}] // Timing
```

40.3880752

40.9088240

41.8501776

43.4725104

46.1563696

{10.274 Second, Null}

```
sol
```

$\{c'[1] \to 0.116233,\ c''[1] \to -0.0357926,\ c^{(3)}[1] \to 0.0541129,\ c^{(4)}[1] \to -0.135759,$
$c^{(5)}[1] \to 0.47631,\ c^{(6)}[1] \to -2.14781,\ c^{(7)}[1] \to 11.8364,\ c^{(8)}[1] \to -77.0954,$
$c^{(9)}[1] \to 579.493,\ c^{(10)}[1] \to -4937.49,\ c^{(11)}[1] \to 47028.,\ c^{(12)}[1] \to -495179.,$
$c^{(13)}[1] \to 5.71168\times 10^6,\ c^{(14)}[1] \to -7.16244\times 10^7,\ c^{(15)}[1] \to 9.70199\times 10^8\}$

```
csol[x_] = Series[c[x], {x, 1, 15}] //. sol // Normal
```

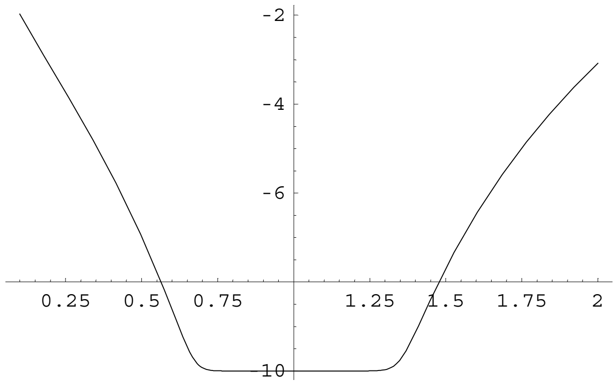
$0.210526 + 0.116233\ (-1+x) - 0.0178963\ (-1+x)^2 + 0.00901882\ (-1+x)^3 -$
$0.00565664\ (-1+x)^4 + 0.00396925\ (-1+x)^5 - 0.00298307\ (-1+x)^6 + 0.0023485\ (-1+x)^7 -$
$0.00191209\ (-1+x)^8 + 0.00159693\ (-1+x)^9 - 0.00136064\ (-1+x)^{10} + 0.00117815\ (-1+x)^{11} -$
$0.00103377\ (-1+x)^{12} + 0.000917241\ (-1+x)^{13} - 0.000821585\ (-1+x)^{14} + 0.000741927\ (-1+x)^{15}$

```
EulerEq
```

$$\frac{1}{c[k]} - \frac{0.95\left(1 + \frac{0.0526316}{(0.210526\, k^{0.25}+k-c[k])^{0.75}}\right)}{c[0.210526\, k^{0.25} + k - c[k]]}$$

```
EulerEqsol = csol[k] EulerEq //. c → csol;

Plot[Log[10, Abs[EulerEqsol] + 10^-10], {k, .1, 2.0}, PlotRange → All];
```



Deterministic Model - abstract

■ setup

We use arbitrary u[c] and f[x]. This will show where various derivatives arise.

```
kplus = f[k] - c[k]
cplus = c[kplus]
fp[x_] = D[f[x], x]
EulerEq = u'[c[k]] - β u'[cplus] fp[kplus]
```

```
-c[k] + f[k]

c[-c[k] + f[k]]

f′[x]

u′[c[k]] - β f′[-c[k] + f[k]] u′[c[-c[k] + f[k]]]
```

```
ss = {k → 1, 1. → 1, 0. → 0}
```

```
{k → 1, 1. → 1, 0. → 0}
```

```
EulerEq //. ss
```

```
u′[c[1]] - β f′[-c[1] + f[1]] u′[c[-c[1] + f[1]]]
```

```
c[1] = f[1] - 1
```

$-1 + f[1]$

```
f[1] = css + 1;
```

```
EulerEq //. ss // Simplify
```

$(1 - \beta f'[1])\, u'[css]$

```
f'[1] = 1 / β
```

$\frac{1}{\beta}$

```
EulerEq //. ss // Simplify
```

0

```
sol = {};
```

k pert

```
D[EulerEq, k]
```

$$-\beta\,(-c'[k] + f'[k])\, u'[c[-c[k] + f[k]]]\, f''[-c[k] + f[k]] + c'[k]\, u''[c[k]] - \beta\, c'[-c[k] + f[k]]\,(-c'[k] + f'[k])\, f'[-c[k] + f[k]]\, u''[c[-c[k] + f[k]]]$$

```
% //. ss
```

$$-\beta\left(\frac{1}{\beta} - c'[1]\right) u'[css]\, f''[1] + c'[1]\, u''[css] - \left(\frac{1}{\beta} - c'[1]\right) c'[1]\, u''[css]$$

```
% // Simplify
```

$$\frac{\beta\,(-1 + \beta\, c'[1])\, u'[css]\, f''[1] + c'[1]\,(-1 + \beta + \beta\, c'[1])\, u''[css]}{\beta}$$

```
Solve[% == 0, c'[1]]
```

$$\left\{\left\{c'[1] \to \frac{1}{2\,\beta\, u''[css]}\left(-\beta^2\, u'[css]\, f''[1] + u''[css] - \beta\, u''[css] - \sqrt{4\,\beta^2\, u'[css]\, f''[1]\, u''[css] + (\beta^2\, u'[css]\, f''[1] - u''[css] + \beta\, u''[css])^2}\right)\right\},\right.$$
$$\left.\left\{c'[1] \to \frac{1}{2\,\beta\, u''[css]}\left(-\beta^2\, u'[css]\, f''[1] + u''[css] - \beta\, u''[css] + \sqrt{4\,\beta^2\, u'[css]\, f''[1]\, u''[css] + (\beta^2\, u'[css]\, f''[1] - u''[css] + \beta\, u''[css])^2}\right)\right\}\right\}$$

```
sol = Union[sol, %[[2]]]
```

$$\left\{c'[1] \to \frac{1}{2\,\beta\, u''[css]}\left(-\beta^2\, u'[css]\, f''[1] + u''[css] - \beta\, u''[css] + \sqrt{4\,\beta^2\, u'[css]\, f''[1]\, u''[css] + (\beta^2\, u'[css]\, f''[1] - u''[css] + \beta\, u''[css])^2}\right)\right\}$$

■ k pert - degree 2

```
D[EulerEq, k, k]
```

$$\begin{aligned}&-\beta\, u'[c[-c[k]+f[k]]]\,(-c''[k]+f''[k])\,f''[-c[k]+f[k]]+c''[k]\,u''[c[k]]-\\&\beta\,(-c'[k]+f'[k])^2\,f'[-c[k]+f[k]]\,c''[-c[k]+f[k]]\,u''[c[-c[k]+f[k]]]-\\&\beta\,c'[-c[k]+f[k]]\,f'[-c[k]+f[k]]\,(-c''[k]+f''[k])\,u''[c[-c[k]+f[k]]]-\\&2\,\beta\,c'[-c[k]+f[k]]\,(-c'[k]+f'[k])^2\,f''[-c[k]+f[k]]\,u''[c[-c[k]+f[k]]]-\\&\beta\,(-c'[k]+f'[k])^2\,u'[c[-c[k]+f[k]]]\,f^{(3)}[-c[k]+f[k]]+c'[k]^2\,u^{(3)}[c[k]]-\\&\beta\,c'[-c[k]+f[k]]^2\,(-c'[k]+f'[k])^2\,f'[-c[k]+f[k]]\,u^{(3)}[c[-c[k]+f[k]]]\end{aligned}$$

```
% //. ss
```

$$\begin{aligned}&-\beta\, u'[css]\,f''[1]\,(-c''[1]+f''[1])+c''[1]\,u''[css]-\left(\frac{1}{\beta}-c'[1]\right)^2 c''[1]\,u''[css]-\\&2\,\beta\left(\frac{1}{\beta}-c'[1]\right)^2 c'[1]\,f''[1]\,u''[css]-c'[1]\,(-c''[1]+f''[1])\,u''[css]-\\&\beta\left(\frac{1}{\beta}-c'[1]\right)^2 u'[css]\,f^{(3)}[1]+c'[1]^2\,u^{(3)}[css]-\left(\frac{1}{\beta}-c'[1]\right)^2 c'[1]^2\,u^{(3)}[css]\end{aligned}$$

```
Solve[% == 0, c''[1]]
```

$$\begin{aligned}&\{\{c''[1]\to(\beta^3\,u'[css]\,f''[1]^2+2\,\beta\,c'[1]\,f''[1]\,u''[css]+\beta^2\,c'[1]\,f''[1]\,u''[css]-\\&\quad 4\,\beta^2\,c'[1]^2\,f''[1]\,u''[css]+2\,\beta^3\,c'[1]^3\,f''[1]\,u''[css]+\beta\,u'[css]\,f^{(3)}[1]-\\&\quad 2\,\beta^2\,c'[1]\,u'[css]\,f^{(3)}[1]+\beta^3\,c'[1]^2\,u'[css]\,f^{(3)}[1]+c'[1]^2\,u^{(3)}[css]-\\&\quad \beta^2\,c'[1]^2\,u^{(3)}[css]-2\,\beta\,c'[1]^3\,u^{(3)}[css]+\beta^2\,c'[1]^4\,u^{(3)}[css])\,/\,(\beta^3\,u'[css]\,f''[1]-\\&\quad u''[css]+\beta^2\,u''[css]+2\,\beta\,c'[1]\,u''[css]+\beta^2\,c'[1]\,u''[css]-\beta^2\,c'[1]^2\,u''[css])\}\}\end{aligned}$$

```
sol = Union[sol, %[[1]]] // Simplify
```

$$\begin{aligned}&\Big\{c'[1]\to\frac{1}{2\,\beta\,u''[css]}\Big(-\beta^2\,u'[css]\,f''[1]+u''[css]-\beta\,u''[css]+\\&\quad\sqrt{4\,\beta^2\,u'[css]\,f''[1]\,u''[css]+(\beta^2\,u'[css]\,f''[1]+(-1+\beta)\,u''[css])^2}\Big),\\&c''[1]\to(\beta\,u'[css]\,(f^{(3)}[1]-2\,\beta\,c'[1]\,f^{(3)}[1]+\beta^2\,(f''[1]^2+c'[1]^2\,f^{(3)}[1]))+\\&\quad c'[1]\,(\beta\,(2+\beta-4\,\beta\,c'[1]+2\,\beta^2\,c'[1]^2)\,f''[1]\,u''[css]+\\&\qquad c'[1]\,(1-2\,\beta\,c'[1]+\beta^2\,(-1+c'[1]^2))\,u^{(3)}[css]))\,/\\&\quad(\beta^3\,u'[css]\,f''[1]+(-1+2\,\beta\,c'[1]+\beta^2\,(1+c'[1]-c'[1]^2))\,u''[css])\Big\}\end{aligned}$$

■ k pert - degree 3

```
D[EulerEq, k, k, k] //. ss
```

$$\begin{aligned}&-3\beta\left(\frac{1}{\beta}-c'[1]\right)^3 c''[1]\,f''[1]\,u''[css]-3\left(\frac{1}{\beta}-c'[1]\right)c''[1]\,(-c''[1]+f''[1])\,u''[css]-\\&6\beta\left(\frac{1}{\beta}-c'[1]\right)c'[1]\,f''[1]\,(-c''[1]+f''[1])\,u''[css]+u''[css]\,c^{(3)}[1]-\\&\left(\frac{1}{\beta}-c'[1]\right)^3 u''[css]\,c^{(3)}[1]-3\beta\left(\frac{1}{\beta}-c'[1]\right)u'[css]\,(-c''[1]+f''[1])\,f^{(3)}[1]-\\&3\beta\left(\frac{1}{\beta}-c'[1]\right)^3 c'[1]\,u''[css]\,f^{(3)}[1]-\beta\,u'[css]\,f''[1]\,(-c^{(3)}[1]+f^{(3)}[1])-\\&c'[1]\,u''[css]\,(-c^{(3)}[1]+f^{(3)}[1])+3\,c'[1]\,c''[1]\,u^{(3)}[css]-\\&3\left(\frac{1}{\beta}-c'[1]\right)^3 c'[1]\,c''[1]\,u^{(3)}[css]-3\beta\left(\frac{1}{\beta}-c'[1]\right)^3 c'[1]^2\,f''[1]\,u^{(3)}[css]-\\&3\left(\frac{1}{\beta}-c'[1]\right)c'[1]^2\,(-c''[1]+f''[1])\,u^{(3)}[css]-\\&\beta\left(\frac{1}{\beta}-c'[1]\right)^3 u'[css]\,f^{(4)}[1]+c'[1]^3\,u^{(4)}[css]-\left(\frac{1}{\beta}-c'[1]\right)^3 c'[1]^3\,u^{(4)}[css]\end{aligned}$$

```
Solve[% == 0, c'''[1]];
```

```
sol = Union[sol, %[[1]]] // Simplify;
```

```
sol
```

$$\begin{aligned}&\Big\{c'[1]\to\frac{1}{2\beta\,u''[css]}\Big(-\beta^2 u'[css]\,f''[1]+u''[css]-\beta\,u''[css]+\\&\quad\sqrt{4\beta^2 u'[css]\,f''[1]\,u''[css]+(\beta^2 u'[css]\,f''[1]+(-1+\beta)\,u''[css])^2}\Big),\\&c''[1]\to(\beta\,u'[css]\,(f^{(3)}[1]-2\beta\,c'[1]\,f^{(3)}[1]+\beta^2\,(f''[1]^2+c'[1]^2\,f^{(3)}[1]))+\\&\quad c'[1]\,(\beta\,(2+\beta-4\beta\,c'[1]+2\beta^2\,c'[1]^2)\,f''[1]\,u''[css]+\\&\qquad c'[1]\,(1-2\beta\,c'[1]+\beta^2\,(-1+c'[1]^2))\,u^{(3)}[css]))\,/\\&\quad(\beta^3 u'[css]\,f''[1]+(-1+2\beta\,c'[1]+\beta^2\,(1+c'[1]-c'[1]^2))\,u''[css]),\ c^{(3)}[1]\to\\&\quad\Big(\beta^3\Big(\frac{3\,(-1+\beta\,c'[1])\,c''[1]\,(c''[1]-f''[1])\,u''[css]}{\beta}-\frac{3\,(-1+\beta\,c'[1])^3\,c''[1]\,f''[1]\,u''[css]}{\beta^2}-\\&\qquad 6\,c'[1]\,(-1+\beta\,c'[1])\,f''[1]\,(-c''[1]+f''[1])\,u''[css]+\\&\qquad 3\,(-1+\beta\,c'[1])\,u'[css]\,(c''[1]-f''[1])\,f^{(3)}[1]+\beta\,u'[css]\,f''[1]\,f^{(3)}[1]+\\&\qquad c'[1]\,u''[css]\,f^{(3)}[1]-\frac{3\,c'[1]\,(-1+\beta\,c'[1])^3\,u''[css]\,f^{(3)}[1]}{\beta^2}-\\&\qquad 3\,c'[1]\,c''[1]\,u^{(3)}[css]-\frac{3\,c'[1]\,(-1+\beta\,c'[1])^3\,c''[1]\,u^{(3)}[css]}{\beta^3}+\\&\qquad\frac{3\,c'[1]^2\,(-1+\beta\,c'[1])\,(c''[1]-f''[1])\,u^{(3)}[css]}{\beta}-\\&\qquad\frac{3\,c'[1]^2\,(-1+\beta\,c'[1])^3\,f''[1]\,u^{(3)}[css]}{\beta^2}-\frac{(-1+\beta\,c'[1])^3\,u'[css]\,f^{(4)}[1]}{\beta^2}-\\&\qquad c'[1]^3\,u^{(4)}[css]-\frac{c'[1]^3\,(-1+\beta\,c'[1])^3\,u^{(4)}[css]}{\beta^3}\Big)\Big)\Big/\\&\quad(\beta^4 u'[css]\,f''[1]+(-1+3\beta\,c'[1]-3\beta^2\,c'[1]^2+\beta^3\,(1+c'[1]+c'[1]^3))\,u''[css])\Big\}\end{aligned}$$

■ k pert - more

```
Do[Print[{i, SessionTime[]}];
  dκ = D[EulerEq, {k, i}] //. ss;
  solκ = Solve[dκ == 0, Derivative[i][c][1]];
   sol = Union[sol, solκ[[1]]] // Simplify,
  {i, 4, 6}] // Timing
```

{4, 1.2618144}

{5, 3.0343632}

{6, 5.7282368}

{13.68 Second, Null}

Stochastic Model

■ setup

```
kplus = f[k, θ] - c[k, θ, ϵ]
cplus = c[kplus, θplus, ϵ]
θplus = ρ θ + ϵ z
fp[x_, y_] = D[f[x, y], x]
EulerEq = u'[c[k, θ, ϵ]] - β u'[cplus] fp[kplus, θplus]
```

−c[k, θ, ϵ] + f[k, θ]

c[−c[k, θ, ϵ] + f[k, θ], θplus, ϵ]

z ϵ + θ ρ

$f^{(1,0)}[x, y]$

$u'[c[k, \theta, \epsilon]] - \beta u'[c[-c[k, \theta, \epsilon] + f[k, \theta], z\epsilon + \theta\rho, \epsilon]]\, f^{(1,0)}[-c[k, \theta, \epsilon] + f[k, \theta], z\epsilon + \theta\rho]$

```
u[x_] = Log[x]
f[x_, y_] = x + A x^α Exp[y]
```

Log[x]

$x + A e^{y} x^{\alpha}$

```
ss = {k → 1, θ → 0, ϵ → 0, 1. → 1, 0. → 0}
```

{k → 1, θ → 0, ϵ → 0, 1. → 1, 0. → 0}

EulerEq //. ss

$$\frac{1}{c[1,\ 0,\ 0]} - \frac{\beta\ (1 + A\,\alpha\ (1 + A - c[1,\ 0,\ 0])^{-1+\alpha})}{c[1 + A - c[1,\ 0,\ 0],\ 0,\ 0]}$$

fp[1, 0]

$1 + A\,\alpha$

Solve[% == 1 / β, A]

$$\left\{\left\{A \to \frac{-1 + \frac{1}{\beta}}{\alpha}\right\}\right\}$$

A = A //. %[[1]]

$$\frac{-1 + \frac{1}{\beta}}{\alpha}$$

α = 1. / 3; β = 95. / 100; ρ = 1. / 2;

EulerEq

$$\frac{1}{c[k,\ \theta,\ \epsilon]} - \frac{0.95\left(1 + \frac{0.0526316\ e^{z\,\epsilon + 0.5\,\theta}}{(0.157895\ e^{\theta}\ k^{0.333333} + k - c[k,\theta,\epsilon])^{0.666667}}\right)}{c[0.157895\ e^{\theta}\ k^{0.333333} + k - c[k,\ \theta,\ \epsilon],\ z\,\epsilon + 0.5\,\theta,\ \epsilon]}$$

EulerEq //. ss

$$\frac{1}{c[1,\ 0,\ 0]} - \frac{0.95\left(1 + \frac{0.0526316}{(1.15789 - c[1,0,0])^{0.666667}}\right)}{c[1.15789 - c[1,\ 0,\ 0],\ 0,\ 0]}$$

c[1, 0, 0] = css = f[1, 0] - 1

0.157895

EulerEq //. ss // Simplify

8.88178×10^{-16}

sol = {};

■ k pert

D[EulerEq, k] //. ss // Simplify

$-40.1111\ (-0.101794 + c^{(1,0,0)}[1,\ 0,\ 0])\ (0.0544254 + c^{(1,0,0)}[1,\ 0,\ 0])$

Solve[% == 0, c^(1,0,0)[1, 0, 0]]

$\{\{c^{(1,0,0)}[1,\ 0,\ 0] \to -0.0544254\},\ \{c^{(1,0,0)}[1,\ 0,\ 0] \to 0.101794\}\}$

sol = Union[sol, %[[2]]]

$\{c^{(1,0,0)}[1,\ 0,\ 0] \to 0.101794\}$

■ θ pert

```
D[EulerEq, θ] //. ss //. sol // Simplify
```

$0.519694 - 24.3497\, c^{(0,1,0)}[1, 0, 0]$

```
Solve[% == 0, c^(0,1,0)[1, 0, 0]]
```

$\{\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429\}\}$

```
sol = Union[sol, %[[1]]]
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(1,0,0)}[1, 0, 0] \to 0.101794\}$

■ k, θ pert - degree 2

```
D[EulerEq, k, k] //. ss //. sol // Simplify
```

$-0.209957 - 8.14113\, c^{(2,0,0)}[1, 0, 0]$

```
Solve[% == 0, c^(2,0,0)[1, 0, 0]]
```

$\{\{c^{(2,0,0)}[1, 0, 0] \to -0.0257897\}\}$

```
sol = Union[sol, %[[1]]] // Simplify
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(1,0,0)}[1, 0, 0] \to 0.101794,\ c^{(2,0,0)}[1, 0, 0] \to -0.0257897\}$

```
D[EulerEq, k, θ] //. ss //. sol // Simplify
```

$0.0900211 - 25.3357\, c^{(1,1,0)}[1, 0, 0]$

```
Solve[% == 0, c^(1,1,0)[1, 0, 0]]
```

$\{\{c^{(1,1,0)}[1, 0, 0] \to 0.00355313\}\}$

```
sol = Union[sol, %[[1]]] // Simplify
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(1,0,0)}[1, 0, 0] \to 0.101794,$
$c^{(1,1,0)}[1, 0, 0] \to 0.00355313,\ c^{(2,0,0)}[1, 0, 0] \to -0.0257897\}$

```
D[EulerEq, θ, θ] //. ss //. sol // Simplify
```

$0.586289 - 34.3775\, c^{(0,2,0)}[1, 0, 0]$

```
Solve[% == 0, c^(0,2,0)[1, 0, 0]]
```

$\{\{c^{(0,2,0)}[1, 0, 0] \to 0.0170544\}\}$

```
sol = Union[sol, %[[1]]] // Simplify
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(0,2,0)}[1, 0, 0] \to 0.0170544,$
$c^{(1,0,0)}[1, 0, 0] \to 0.101794,\ c^{(1,1,0)}[1, 0, 0] \to 0.00355313,\ c^{(2,0,0)}[1, 0, 0] \to -0.0257897\}$

■ k,θ pert - degree 3

```
D[EulerEq, k, k, k] //. ss //. sol // Simplify
```

$0.37707 - 9.92396\, c^{(3,0,0)}[1, 0, 0]$

```
Solve[% == 0, c^(3,0,0)[1, 0, 0]]
```

$\{\{c^{(3,0,0)}[1, 0, 0] \to 0.0379959\}\}$

```
sol = Union[sol, %[[1]]] // Simplify;
```

```
D[EulerEq, k, k, θ] //. ss //. sol // Simplify
```

$-0.0642187 - 26.2732\, c^{(2,1,0)}[1, 0, 0]$

```
Solve[% == 0, c^(2,1,0)[1, 0, 0]]
```

$\{\{c^{(2,1,0)}[1, 0, 0] \to -0.00244427\}\}$

```
sol = Union[sol, %[[1]]] // Simplify
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(0,2,0)}[1, 0, 0] \to 0.0170544,$
$c^{(1,0,0)}[1, 0, 0] \to 0.101794,\ c^{(1,1,0)}[1, 0, 0] \to 0.00355313,\ c^{(2,0,0)}[1, 0, 0] \to -0.0257897,$
$c^{(2,1,0)}[1, 0, 0] \to -0.00244427,\ c^{(3,0,0)}[1, 0, 0] \to 0.0379959\}$

```
D[EulerEq, k, θ, θ] //. ss //. sol // Simplify
```

$0.0904829 - 34.8705\, c^{(1,2,0)}[1, 0, 0]$

```
Solve[% == 0, c^(1,2,0)[1, 0, 0]]
```

$\{\{c^{(1,2,0)}[1, 0, 0] \to 0.00259483\}\}$

```
sol = Union[sol, %[[1]]] // Simplify;
```

```
D[EulerEq, θ, θ, θ] //. ss //. sol // Simplify
```

$0.606153 - 39.3914\, c^{(0,3,0)}[1, 0, 0]$

```
Solve[% == 0, c^(0,3,0)[1, 0, 0]]
```

$\{\{c^{(0,3,0)}[1, 0, 0] \to 0.015388\}\}$

```
sol = Union[sol, %[[1]]] // Simplify
```

$\{c^{(0,1,0)}[1, 0, 0] \to 0.0213429,\ c^{(0,2,0)}[1, 0, 0] \to 0.0170544,\ c^{(0,3,0)}[1, 0, 0] \to 0.015388,$
$c^{(1,0,0)}[1, 0, 0] \to 0.101794,\ c^{(1,1,0)}[1, 0, 0] \to 0.00355313,\ c^{(1,2,0)}[1, 0, 0] \to 0.00259483,$
$c^{(2,0,0)}[1, 0, 0] \to -0.0257897,\ c^{(2,1,0)}[1, 0, 0] \to -0.00244427,\ c^{(3,0,0)}[1, 0, 0] \to 0.0379959\}$

■ ϵ pert

```
D[EulerEq, ϵ] //. ss //. sol // Simplify
```

```
0.539421 z - 4.29417 c^(0,0,1)[1, 0, 0]
```

```
% //. z → 0
```

```
-4.29417 c^(0,0,1)[1, 0, 0]
```

```
Solve[% == 0, c^(0,0,1)[1, 0, 0]]
```

```
{{c^(0,0,1)[1, 0, 0] → 0.}}
```

```
sol = Union[sol, %[[1]]] // Chop
```

```
{c^(0,0,1)[1, 0, 0] → 0, c^(0,1,0)[1, 0, 0] → 0.0213429, c^(0,2,0)[1, 0, 0] → 0.0170544,
 c^(0,3,0)[1, 0, 0] → 0.015388, c^(1,0,0)[1, 0, 0] → 0.101794, c^(1,1,0)[1, 0, 0] → 0.00355313,
 c^(1,2,0)[1, 0, 0] → 0.00259483, c^(2,0,0)[1, 0, 0] → -0.0257897,
 c^(2,1,0)[1, 0, 0] → -0.00244427, c^(3,0,0)[1, 0, 0] → 0.0379959}
```

```
sol = Union[sol, {c^(i_,j_,1)[1, 0, 0] → 0}];
```

■ ϵ pert - second order

```
D[EulerEq, ϵ, ϵ] //. ss //. sol // Simplify
```

```
0.221577 z^2 - 4.29417 c^(0,0,2)[1, 0, 0]
```

```
% // Expand
```

```
0.221577 z^2 - 4.29417 c^(0,0,2)[1, 0, 0]
```

```
% //. z^2 → σ^2
```

```
0.221577 σ^2 - 4.29417 c^(0,0,2)[1, 0, 0]
```

```
% //. z → 0
```

```
0.221577 σ^2 - 4.29417 c^(0,0,2)[1, 0, 0]
```

```
Solve[% == 0, c^(0,0,2)[1, 0, 0]]
```

```
{{c^(0,0,2)[1, 0, 0] → 0.0515994 σ^2}}
```

```
sol = Union[sol, %[[1]]];
sol = sol // Simplify
```

```
{c^(0,0,1)[1, 0, 0] → 0, c^(0,0,2)[1, 0, 0] → 0.0515994 σ^2, c^(0,1,0)[1, 0, 0] → 0.0213429,
 c^(0,2,0)[1, 0, 0] → 0.0170544, c^(0,3,0)[1, 0, 0] → 0.015388, c^(1,0,0)[1, 0, 0] → 0.101794,
 c^(1,1,0)[1, 0, 0] → 0.00355313, c^(1,2,0)[1, 0, 0] → 0.00259483, c^(2,0,0)[1, 0, 0] → -0.0257897,
 c^(2,1,0)[1, 0, 0] → -0.00244427, c^(3,0,0)[1, 0, 0] → 0.0379959, c^(i_,j_,1)[1, 0, 0] → 0}
```

■ ϵ pert - third order

```
D[EulerEq, ϵ, ϵ, ϵ] //. ss //. sol // Simplify
```

$0.035918\, z^3 - 1.2792\, z\, \sigma^2 - 4.29417\, c^{(0,0,3)}[1,0,0] + 120.333\, z\, c^{(0,1,2)}[1,0,0]$

```
% // Expand
```

$0.035918\, z^3 - 1.2792\, z\, \sigma^2 - 4.29417\, c^{(0,0,3)}[1,0,0] + 120.333\, z\, c^{(0,1,2)}[1,0,0]$

```
% //. z^3 → λ;
```

```
% //. z^2 → σ^2;
```

```
% //. z → 0
```

$0.035918\, \lambda - 4.29417\, c^{(0,0,3)}[1,0,0]$

```
Solve[% == 0, c^(0,0,3)[1, 0, 0]]
```

$\{\{c^{(0,0,3)}[1,0,0] \to 0.00836436\, \lambda\}\}$

```
sol = Union[sol, %[[1]]];
sol = sol // Simplify
```

$\{c^{(0,0,1)}[1,0,0] \to 0,\ c^{(0,0,2)}[1,0,0] \to 0.0515994\, \sigma^2,\ c^{(0,0,3)}[1,0,0] \to 0.00836436\, \lambda,$
$c^{(0,1,0)}[1,0,0] \to 0.0213429,\ c^{(0,2,0)}[1,0,0] \to 0.0170544,$
$c^{(0,3,0)}[1,0,0] \to 0.015388,\ c^{(1,0,0)}[1,0,0] \to 0.101794,\ c^{(1,1,0)}[1,0,0] \to 0.00355313,$
$c^{(1,2,0)}[1,0,0] \to 0.00259483,\ c^{(2,0,0)}[1,0,0] \to -0.0257897,$
$c^{(2,1,0)}[1,0,0] \to -0.00244427,\ c^{(3,0,0)}[1,0,0] \to 0.0379959,\ c^{(i\_,j\_,1)}[1,0,0] \to 0\}$

■ (ϵ,ϵ,k) pert

```
D[EulerEq, ϵ, ϵ, k] //. ss //. sol // Simplify
```

$-0.152533\, z^2 + 0.405267\, \sigma^2 - 6.26612\, c^{(1,0,2)}[1,0,0]$

```
% // Expand
```

$-0.152533\, z^2 + 0.405267\, \sigma^2 - 6.26612\, c^{(1,0,2)}[1,0,0]$

```
% //. z^3 → λ;
```

```
% //. z^2 → σ^2;
```

```
% //. z → 0
```

$0.252734\, \sigma^2 - 6.26612\, c^{(1,0,2)}[1,0,0]$

```
Solve[% == 0, c^(1,0,2)[1, 0, 0]]
```

$\{\{c^{(1,0,2)}[1,0,0] \to 0.0403334\, \sigma^2\}\}$

```
sol = Union[sol, %[[1]]];
sol = sol // Simplify
```

$\{c^{(0,0,1)}[1,0,0] \to 0,\ c^{(0,0,2)}[1,0,0] \to 0.0515994\,\sigma^2,\ c^{(0,0,3)}[1,0,0] \to 0.00836436\,\lambda,\ c^{(0,1,0)}[1,0,0] \to 0.0213429,\ c^{(0,2,0)}[1,0,0] \to 0.0170544,\ c^{(0,3,0)}[1,0,0] \to 0.015388,\ c^{(1,0,0)}[1,0,0] \to 0.101794,\ c^{(1,0,2)}[1,0,0] \to 0.0403334\,\sigma^2,\ c^{(1,1,0)}[1,0,0] \to 0.00355313,\ c^{(1,2,0)}[1,0,0] \to 0.00259483,\ c^{(2,0,0)}[1,0,0] \to -0.0257897,\ c^{(2,1,0)}[1,0,0] \to -0.00244427,\ c^{(3,0,0)}[1,0,0] \to 0.0379959,\ c^{(i\_,j\_,1)}[1,0,0] \to 0\}$

■ (ϵ,ϵ,θ) pert

```
D[EulerEq, ϵ, ϵ, θ] //. ss //. sol // Simplify
```

$-0.00394657\,z^2 + 0.242199\,\sigma^2 - 24.3497\,c^{(0,1,2)}[1,0,0]$

```
% // Expand
```

$-0.00394657\,z^2 + 0.242199\,\sigma^2 - 24.3497\,c^{(0,1,2)}[1,0,0]$

```
% //. z^3 → λ;
```

```
% //. z^2 → σ^2;
```

```
% //. z → 0
```

$0.238252\,\sigma^2 - 24.3497\,c^{(0,1,2)}[1,0,0]$

```
Solve[% == 0, c^(0,1,2)[1, 0, 0]]
```

$\{\{c^{(0,1,2)}[1,0,0] \to 0.00978459\,\sigma^2\}\}$

```
sol = Union[sol, %[[1]]];
sol = sol // Simplify
```

$\{c^{(0,0,1)}[1,0,0] \to 0,\ c^{(0,0,2)}[1,0,0] \to 0.0515994\,\sigma^2,\ c^{(0,0,3)}[1,0,0] \to 0.00836436\,\lambda,\ c^{(0,1,0)}[1,0,0] \to 0.0213429,\ c^{(0,1,2)}[1,0,0] \to 0.00978459\,\sigma^2,\ c^{(0,2,0)}[1,0,0] \to 0.0170544,\ c^{(0,3,0)}[1,0,0] \to 0.015388,\ c^{(1,0,0)}[1,0,0] \to 0.101794,\ c^{(1,0,2)}[1,0,0] \to 0.0403334\,\sigma^2,\ c^{(1,1,0)}[1,0,0] \to 0.00355313,\ c^{(1,2,0)}[1,0,0] \to 0.00259483,\ c^{(2,0,0)}[1,0,0] \to -0.0257897,\ c^{(2,1,0)}[1,0,0] \to -0.00244427,\ c^{(3,0,0)}[1,0,0] \to 0.0379959,\ c^{(i\_,j\_,1)}[1,0,0] \to 0\}$

Approximation

```
Series[c[1 + δ k , δ θ, δ ϵ], {δ, 0, 3}]
```

$$0.157895 + (\epsilon\, c^{(0,0,1)}[1,0,0] + \theta\, c^{(0,1,0)}[1,0,0] + k\, c^{(1,0,0)}[1,0,0])\,\delta + \left(\frac{1}{2}\epsilon^2 c^{(0,0,2)}[1,0,0] + \epsilon\,\theta\, c^{(0,1,1)}[1,0,0] + \frac{1}{2}\theta^2 c^{(0,2,0)}[1,0,0] + k\,\epsilon\, c^{(1,0,1)}[1,0,0] + k\,\theta\, c^{(1,1,0)}[1,0,0] + \frac{1}{2}k^2 c^{(2,0,0)}[1,0,0]\right)\delta^2 + \left(\frac{1}{6}\epsilon^3 c^{(0,0,3)}[1,0,0] + \frac{1}{2}\epsilon^2\theta\, c^{(0,1,2)}[1,0,0] + \frac{1}{2}\epsilon\,\theta^2 c^{(0,2,1)}[1,0,0] + \frac{1}{6}\theta^3 c^{(0,3,0)}[1,0,0] + \frac{1}{2}k\,\epsilon^2 c^{(1,0,2)}[1,0,0] + k\,\epsilon\,\theta\, c^{(1,1,1)}[1,0,0] + \frac{1}{2}k\,\theta^2 c^{(1,2,0)}[1,0,0] + \frac{1}{2}k^2\epsilon\, c^{(2,0,1)}[1,0,0] + \frac{1}{2}k^2\theta\, c^{(2,1,0)}[1,0,0] + \frac{1}{6}k^3 c^{(3,0,0)}[1,0,0]\right)\delta^3 + O[\delta]^4$$

```
% //. sol
```

$$0.157895 + (0.101794\,k + 0.0213429\,\theta)\,\delta + (-0.0128949\,k^2 + 0.00355313\,k\,\theta + 0.00852722\,\theta^2 + 0.0257997\,\epsilon^2\sigma^2)\,\delta^2 + (0.00633266\,k^3 - 0.00122213\,k^2\theta + 0.00129741\,k\,\theta^2 + 0.00256466\,\theta^3 + 0.00139406\,\epsilon^3\lambda + 0.0201667\,k\,\epsilon^2\sigma^2 + 0.0048923\,\epsilon^2\theta\,\sigma^2)\,\delta^3 + O[\delta]^4$$

```
% // Normal //. δ → 1
```

$$0.157895 + \delta\,(0.101794\,k + 0.0213429\,\theta) + \delta^2(-0.0128949\,k^2 + 0.00355313\,k\,\theta + 0.00852722\,\theta^2 + 0.0257997\,\epsilon^2\sigma^2) + \delta^3(0.00633266\,k^3 - 0.00122213\,k^2\theta + 0.00129741\,k\,\theta^2 + 0.00256466\,\theta^3 + 0.00139406\,\epsilon^3\lambda + 0.0201667\,k\,\epsilon^2\sigma^2 + 0.0048923\,\epsilon^2\theta\,\sigma^2)$$