PERTURBATION METHODS

Kenneth L. Judd
Hoover Institution and NBER

June 28, 2006
Local Approximation Methods

- Use information about $f : R \to R$ only at a point, $x_0 \in R$, to construct an approximation valid near $x_0$.

- Taylor Series Approximation

\[
f(x) \approx f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + O(|x - x_0|^{n+1})
\]

\[= p_n(x) + O(|x - x_0|^{n+1})\]

- Power series: $\sum_{n=0}^{\infty} a_n z^n$

  - The radius of convergence is

\[r = \sup\{|z| : \sum_{n=0}^{\infty} a_n z^n < \infty\},\]

  - $\sum_{n=0}^{\infty} a_n z^n$ converges for all $|z| < r$ and diverges for all $|z| > r$.

- Complex analysis

  - $f : \Omega \subset C \to C$ on the complex plane $C$ is analytic on $\Omega$ iff

\[\forall a \in \Omega \ \exists r, c_k \left( \forall ||z - a|| < r \left( f(z) = \sum_{k=0}^{\infty} c_k(z - a)^k \right) \right)\]

  - A singularity of $f$ is any $a$ s. t. $f$ is analytic on $\Omega - \{a\}$ but not on $\Omega$.

  - If $f$ or any derivative of $f$ has a singularity at $z \in C$, then the radius of convergence in $C$ of

\[\sum_{n=0}^{\infty} \frac{(x-x_0)^n}{n!} f^{(n)}(x_0),\]

is bounded above by $||x_0 - z||$. 
• Example:  \( f(x) = x^\alpha \) where \( 0 < \alpha < 1 \).

  – One singularity at \( x = 0 \)
  – Radius of convergence for power series around \( x = 1 \) is 1.
  – Taylor series coefficients decline slowly:

\[
a_k = \frac{1}{k!} \frac{d^k}{dx^k} (x^\alpha)|_{x=1} = \frac{\alpha(\alpha-1) \cdots (\alpha-k+1)}{1 \cdot 2 \cdots k}.
\]

Table 6.1 (corrected): Taylor Series Approximation Errors for \( x^{1/4} \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>N: 5</th>
<th>10</th>
<th>20</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>5((-1))</td>
<td>8(1)</td>
<td>3(3)</td>
<td>1(12)</td>
</tr>
<tr>
<td>2.0</td>
<td>1((-2))</td>
<td>5((-3))</td>
<td>2((-3))</td>
<td>8((-4))</td>
</tr>
<tr>
<td>1.8</td>
<td>4((-3))</td>
<td>5((-4))</td>
<td>2((-4))</td>
<td>9((-9))</td>
</tr>
<tr>
<td>1.5</td>
<td>2((-4))</td>
<td>3((-6))</td>
<td>1((-9))</td>
<td>0((-12))</td>
</tr>
<tr>
<td>1.2</td>
<td>1((-6))</td>
<td>2((-10))</td>
<td>0((-12))</td>
<td>0((-12))</td>
</tr>
<tr>
<td>.80</td>
<td>2((-6))</td>
<td>3((-10))</td>
<td>0((-12))</td>
<td>0((-12))</td>
</tr>
<tr>
<td>.50</td>
<td>6((-4))</td>
<td>9((-6))</td>
<td>4((-9))</td>
<td>0((-12))</td>
</tr>
<tr>
<td>.25</td>
<td>1((-2))</td>
<td>1((-3))</td>
<td>4((-5))</td>
<td>3((-9))</td>
</tr>
<tr>
<td>.10</td>
<td>6((-2))</td>
<td>2((-2))</td>
<td>4((-3))</td>
<td>6((-5))</td>
</tr>
<tr>
<td>.05</td>
<td>1((-1))</td>
<td>5((-2))</td>
<td>2((-2))</td>
<td>2((-3))</td>
</tr>
</tbody>
</table>
Log-Linearization and General Nonlinear COV

- Implicit differentiation implies

\[ \dot{x} = \frac{dx}{x} = -\frac{\varepsilon f_{\varepsilon}}{x f_x} \varepsilon = -\frac{\varepsilon f_{\varepsilon}}{x f_x} \varepsilon, \]

- Since \( \dot{x} = d(\ln x) \), log-linearization implies log-linear approximation

\[ \ln x - \ln x_0 = -\frac{\varepsilon_0 f_{\varepsilon}(x_0, \varepsilon_0)}{x_0 f_x(x_0, \varepsilon_0)}(\ln \varepsilon - \ln \varepsilon_0). \] \hspace{1cm} (6.1.5)

- Generalization to nonlinear change of variables.

  - Take any monotonic \( h(\cdot) \), and define \( x = h(X) \) and \( y = h(Y) \)
  - Use the identity

\[ f(Y, X) = f(h^{-1}(h(Y)), h^{-1}(h(X))) = f(h^{-1}(y), h^{-1}(x)) \equiv g(y, x). \]

  to generate expansions

\[ y(x) = y(x_0) + y'(x)(x - x_0) + \ldots \]

\[ Y(X) = h^{-1}\left(y(h(X_0)) + y'(h(X_0))(h(X) - h(X_0)) \right) + \ldots \]

- \( h(z) = \ln z \) is commonly used by economists, but others may be better globally
Implicit Function Theorem

• Suppose $h : \mathbb{R}^n \to \mathbb{R}^m$ is defined in $H(x, h(x)) = 0$, $H : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$, and $h(x_0) = y_0$.

  - Implicit differentiation shows
    $$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$

  - At $x = x_0$, this implies
    $$h_x(x_0) = -H_y(x_0, y_0)^{-1}H_x(x_0, y_0)$$

    if $H_y(x_0, y_0)$ is nonsingular. More simply, we express this as
    $$h^0_x = -(H^0_y)^{-1}H^0_x$$

  - Linear approximation for $h(x)$ is
    $$h^L(x) = h(x_0) + h_x(x_0)(x - x_0)$$

• To check on quality, we compute
  $$E = \hat{H}(x, h^L(x))$$

  where $\hat{H}$ is a unit free equivalent of $H$. If $E \leq \varepsilon$, then we have an $\varepsilon$-solution.
- If \( h^L(y) \) is not satisfactory, compute higher-order terms by repeated differentiation.

  \[ \frac{D_{xx} H(x, h(x))}{0} \] implies

  \[
  H_{xx} + 2H_{xy}h_x + H_{yy}h_xh_x + H_yh_{xx} = 0
  \]

- At \( x = x_0 \), this implies

  \[
  h_{xx}^0 = -\left(H_y^0\right)^{-1} (H_{xx}^0 + 2H_{xy}^0h_x^0 + H_{yy}^0h_x^0h_x^0)
  \]

- Construct the quadratic approximation

  \[
  h^Q(x) \doteq h(x_0) + h_x^0(x - x_0) + \frac{1}{2}(x - x_0)^\top h_{xx}^0(x - x_0)
  \]

  and check its quality by computing \( E = H(x, h^Q(x)) \).
Regular Perturbation: The Basic Idea

- Suppose $x$ is an endogenous variable, $\varepsilon$ a parameter
  - Want to find $x(\varepsilon)$ such that $f(x(\varepsilon), \varepsilon) = 0$
  - Suppose $x(0)$ known.

- Use Implicit Function Theorem
  - Apply implicit differentiation:
    $$f_x(x(\varepsilon), \varepsilon)x'(\varepsilon) + f_\varepsilon(x(\varepsilon), \varepsilon) = 0$$  
    (13.1.5)
  - At $\varepsilon = 0$, $x(0)$ is known and (13.1.5) is linear in $x'(0)$ with solution
    $$x'(0) = -f_x(x(0), 0)^{-1}f_\varepsilon(x(0), 0)$$
  - Well-defined only if $f_x \neq 0$, a condition which can be checked at $x = x(0)$.
  - The linear approximation of $x(\varepsilon)$ for $\varepsilon$ near zero is
    $$x(\varepsilon) \doteq x^L(\varepsilon) \equiv x(0) - f_x(x(0), 0)^{-1}f_\varepsilon(x(0), 0)\varepsilon$$  
    (13.1.6)
• Can continue for higher-order derivatives of $x(\varepsilon)$.

  - Differentiate (13.1.5) w.r.t. $\varepsilon$

    \[ f_{xx}x'' + f_{xx}(x')^2 + 2f_{x\varepsilon}x' + f_{\varepsilon\varepsilon} = 0 \]  

    (13.1.7)

  - At $\varepsilon = 0$, (13.1.7) implies that

    \[ x''(0) = -f_x(x(0), 0)^{-1} (f_{xx}(x(0), 0) (x'(0))^2 \]
    \[ + 2f_{x\varepsilon}(x(0), 0) x'(0) + f_{\varepsilon\varepsilon}(x(0), 0)) \]

  - Quadratic approximation is

    \[ x(\varepsilon) \approx x^Q(\varepsilon) \equiv x(0) + \varepsilon x'(0) + \frac{1}{2} \varepsilon^2 x''(0) \]  

    (13.1.8)
• General Perturbation Strategy

- Find special (likely degenerate, uninteresting) case where one knows solution
  * General relativity theory: begin with case of a universe with zero mass: $\varepsilon$ is mass of universe
  * Quantum mechanics: begin with case where electrons do not repel each other: $\varepsilon$ is force of repulsion
  * Business cycle analysis: begin with case where there are no shocks: $\varepsilon$ is measure of exogenous shocks

- Use local approximation theory to compute nearby cases
  * Standard implicit function may be applicable
  * Sometimes standard implicit function theorem will not apply; use appropriate bifurcation or singularity method.

- Check to see if solution is good for problem of interest
  * Use unit-free formulation of problem
  * Go to higher-order terms until error is reduced to acceptable level
  * Always check solution for range of validity
Single-Sector, Deterministic Growth - canonical problem

- Consider dynamic programming problem

\[
\max_{c(t)} \int_{0}^{\infty} e^{-\rho t} u(c) dt
\]

\[
\dot{k} = f(k) - c
\]

- Ad-Hoc Method: Convert to a wrong LQ problem

  - McGrattan, JBES (1990)
    * Replace \( u(c) \) and \( f(k) \) with approximations around \( c^* \) and \( k^* \)
    * Solve linear-quadratic problem

\[
\max_{c} \int_{0}^{\infty} e^{-\rho t} \left( u(c^*) + u'(c^*)(c - c^*) + \frac{1}{2}u''(c^*)(c - c^*)^2 \right) dt
\]

s.t. \( \dot{k} = f(k^*) + f'(k^*)(k^* - k) - c \)

  * Resulting approximate policy function is

\[
C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)
\]

  * Local approximate law of motion is \( \dot{k} = 0 \); add noise to get

\[
dk = 0 \cdot dt + dz
\]

  * Approximation is random walk when theory says solution is stationary

• Kydland-Prescott
  – Restate problem so that \( \dot{k} \) is linear function of state and controls
  – Replace \( u(c) \) with quadratic approximation
  – Note 1: such transformation may not be easy
  – Note 2: special case of Magill (JET 1977).

• Lesson
  – Kydland-Prescott, McGrattan provide no mathematical basis for method
  – Formal calculations based on appropriate IFT should be used.
  – Beware of \textit{ad hoc} methods based on an intuitive story!
Perturbation Method for Dynamic Programming

- Formalize problem as a system of functional equations

  - Bellman equation:
    \[ \rho V(k) = \max_c u(c) + V'(k)(f(k) - c) \] (1)

  - \(C(k)\): policy function defined by
    \[ 0 = u'(C(k)) - V'(k) \] (2)
    \[ \rho V(k) = u(C(k)) + V'(k)(f(k) - C(k)) \]

  - Apply envelope theorem to (1) to get
    \[ \rho V'(k) = V''(k)(f(k) - C(k)) + V'(k)f'(k) \] (1k)

  - Steady-state equations
    \[ c^* = f(k^*) \quad \quad \quad \rho V(k^*) = u(c^*) + V'(k^*)(f(k^*) - c^*) \]
    \[ 0 = u'(c^*) - V'(k^*) \quad \rho V'(k) = V''(k)(f(k) - c^*) + V'(k)f'(k) \]

  - Steady State: We know \(k^*, V(k^*), C(k^*), f'(k^*), V'(k^*)\):
    \[ \rho = f'(k^*), \quad C(k^*) = f(k^*), \quad V(k^*) = \rho^{-1}u(c^*), \quad V'(k^*) = u'(c^*) \]

  - Want Taylor expansion:
    \[ C(k) = C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ... \]
    \[ V(k) = V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + ... \]
Linear approximation around a steady state

- Differentiate \((1_k, 2)\) w.r.t. \(k\):

\[
\rho V'' = V'''(f - C') + V''(f' - C''(k)) + V''f' + V'f''
\]

\[
0 = u''C' = V''
\]

- At the steady state

\[
0 = -V''(k^*)C''(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*)
\]

- Substituting \((2_k)\) into \((1_k)\) yields

\[
0 = -u''(C'')^2 + u''C'f' + V'f''
\]

- Two solutions

\[
C''(k^*) = \frac{\rho}{2} \left(1 \pm \sqrt{1 + \frac{4u'(C'(k^*))f'''(k^*)}{u''(C''(k^*))f'(k^*)f''(k^*)}}\right)
\]

- However, we know \(C''(k^*) > 0\); hence, take positive solution
Higher-Order Expansions

- Conventional perception in macroeconomics: “perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ...” – Marcet (1994, p. 111)
- Mathematics literature: No problem (See, e.g., Bensoussan, Fleming, Souganides, etc.)

Compute \( C''(k^*) \) and \( V'''(k^*) \).

- Differentiate \((1_{kk}, 2_k)\):

\[
\rho V''' = V'''(f - C) + 2V'''(f' - C') + V''(f'' - C'') + V''' f' + 2V'' f'' + V' f'''
\]

\[
0 = u'''(C')^2 + u'' C'' - V'''
\]

- At \( k^* \), \((1_{kkk})\) reduces to

\[
0 = 2V'''(f' - C') + 3V'' f'' - V'' C'' + V' f'''
\]

- Equations \((1_{kkk}; 2_{kk})\) are LINEAR in unknowns \( C''(k^*) \) and \( V'''(k^*) \):

\[
\begin{pmatrix}
  u'' & -1 \\
  V'' - 2(f' - C')
\end{pmatrix}
\begin{pmatrix}
  C'' \\
  V'''
\end{pmatrix}
= \begin{pmatrix}
  A_1 \\
  A_2
\end{pmatrix}
\]

- Unique solution since determinant \(-2u''(f' - C') + V'' < 0\).
• Compute $C^{(n)}(k^*)$ and $V^{(n+1)}(k^*)$.

  – Linear system for order $n$ is, for some $A_1$ and $A_2$,

    $$
    \begin{pmatrix}
    u'' & -1 \\
    V'' - n(f' - C')
    \end{pmatrix}
    \begin{pmatrix}
    C^{(n)} \\
    V^{(n+1)}
    \end{pmatrix}
    =
    \begin{pmatrix}
    A_1 \\
    A_2
    \end{pmatrix}
    $$

  – Higher-order terms are produced by solving linear systems

  – The linear system is always determinate since $-nu''(f' - C') + V'' < 0$

• Conclusion:

  – Computing first-order terms involves solving quadratic equations

  – Computing higher-order terms involves solving linear equations

  – Computing higher-order terms is easier than computing the linear term.
Accuracy Measure
Consider the one-period relative Euler equation error:

\[ E(k) = 1 - \frac{V'(k)}{u'(C(k))} \]

- Equilibrium requires it to be zero.
- \( E(k) \) is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - \( E(k) \) is unit-free.
- Define the \( L^p, 1 \leq p < \infty \), bounded rationality accuracy to be
  \[ \log_{10} \| E(k) \|_p \]
- The \( L^\infty \) error is the maximum value of \( E(k) \).
Global Quality of Asymptotic Approximations

- Linear approximation is very poor even for \( k \) close to steady state
- Order 2 is better but still not acceptable for even \( k = .9, 1.1 \)
- Order 10 is excellent for \( k \in [.6, 1.4] \)
Stochastic, Discrete-Time Growth

\[
\max_{c_t} \quad E \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) \right\} \\
\text{s.t.} \quad k_{t+1} = (1 + \varepsilon z) F(k_t - c_t)
\]

(13.7.19)

• New state variable:
  
  – \(k_t\) is capital stock at the beginning of period \(t\)
  
  – consumption comes out of \(k\)
  
  – the remaining capital, \(k_t - c_t\), is used in production
  
  – resulting output is \((1 + \varepsilon z) F(k_t - c_t) = k_{t+1}\)
  
  – perturbation parameter is \(\varepsilon\), the standard deviation, not the variance.

• Do deterministic perturbation analysis.
  
  – Solution when \(\varepsilon = 0\) is \(C(k)\) solving

\[
u'(C(k)) = \beta u'(C(F(k - C(k)))) F'(k - C(k)).
\]

(13.7.20)

  – At the steady state, \(k^*\), \(F(k^* - C(k^*)) = k^*\), and \(1 = \beta F'(k^* - C(k^*))\)

  – Derivative of (13.7.20) with respect to \(k\) implies

\[
u''(C(k)) C''(k) = \beta u''(C(F(k - C(k)))) C''(F(k - C(k)))
\times F'(k - C(k)) [1 - C'(k)] F'(k - C(k))
+ \beta u'(C(F(k - C(k)))) F''(k - C(k)) [1 - C''(k)]
\]

(13.7.21)
At \( k = k^* \), (13.7.21) reduces to (drop all arguments)

\[
  u''C' = \beta u''C' F'[1 - C'] F' + \beta u' F''[1 - C'].
\]

(13.7.22)

with stable solution

\[
  C' = \frac{1}{2} \left( 1 - \beta - \beta^2 \frac{u'}{u''} F'' + \sqrt{\left( 1 - \beta - \beta^2 \frac{u'}{u''} F'' \right)^2 + 4 \frac{u'}{u''} \beta^2 F''} \right)
\]

Take another derivative of (13.7.21) and set \( k = k^* \) to find

\[
  u''C'' + u'''C'C' = \beta u''' \left( C' F'(1 - C') \right)^2 F' + \beta u''C'' \left( F'(1 - C') \right)^2 F' \\
  + 2\beta u''C' F'(1 - C')^2 F'' + \beta u' F'''(1 - C')^2 \\
  + \beta u' F''(-C''),
\]

which is a linear equation in the unknown \( C'''(k^*) \).
• Stochastic problem:

- Euler equation is

\[ u' (C(k)) = \beta E \{ u' (g(\varepsilon, k, z)) \cdot R(\varepsilon, k, z) \}, \]  

(13.7.23)

where

\[ g(\varepsilon, k, z) \equiv C((1 + \varepsilon z)F(k - C(k))), \]  

(13.7.24)

\[ R(\varepsilon, k, z) \equiv (1 + \varepsilon z)F'(k - C(k)). \]

- Compute \( C_\varepsilon \)

* Differentiate (13.7.24) with respect to \( \varepsilon \) yields (we drop arguments of \( F \) and \( C \))

\[ g_\varepsilon = C_\varepsilon + C'' ( zF - (1 + \varepsilon z)F'C_\varepsilon ), \]  

(13.7.25)

\[ g_{\varepsilon\varepsilon} = C_{\varepsilon\varepsilon} + 2C'_\varepsilon ( zF - (1 + \varepsilon z)F'C_\varepsilon ) + C''' ( zF - (1 + \varepsilon z)F'C_\varepsilon )^2, \]

\[ + C'' (-2F' C_\varepsilon + F'' C_\varepsilon^2 - (1 + \varepsilon z) F'C_{\varepsilon\varepsilon}). \]

* At \( \varepsilon = 0 \), (13.7.25) implies that

\[ g_\varepsilon = C_\varepsilon + C''( zF - F'C_\varepsilon ), \]  

(13.7.26)

\[ g_{\varepsilon\varepsilon} = C_{\varepsilon\varepsilon} + 2C'_\varepsilon ( zF - F'C_\varepsilon ) + C''' ( zF - F'C_\varepsilon )^2, \]

\[ + C''(-2F' C_\varepsilon + F'' C_\varepsilon^2 - F'C_{\varepsilon\varepsilon}). \]
* Differentiate (13.7.23) with respect to \( \varepsilon \)

\[
\frac{d^2}{d\varepsilon^2} u_{00} C_{\varepsilon} = \beta \mathbb{E} \left\{ \frac{d^2}{d\varepsilon^2} g_{\varepsilon} (1 + \varepsilon z) F' + u' F' z - u'(1 + \varepsilon z) \frac{d^2}{d\varepsilon^2} C_{\varepsilon} \right\} \tag{13.7.27}
\]

\[
\frac{d^3}{d\varepsilon^3} u_{00} C_{\varepsilon}^2 + \frac{d^3}{d\varepsilon^3} u_{00} C_{\varepsilon\varepsilon} = \beta \mathbb{E} \left\{ \frac{d^3}{d\varepsilon^3} g_{\varepsilon}^2 (1 + \varepsilon z) F' + 2u'' g_{\varepsilon} F' z - 2u'' g_{\varepsilon} F' z - 2u' z F'' C_{\varepsilon} + u'(1 + \varepsilon z) F'' C_{\varepsilon}^2 - u'(1 + \varepsilon z) F'' C_{\varepsilon\varepsilon} \right\} \tag{13.7.28}
\]

* Since \( \mathbb{E}\{z\} = 0 \), (13.7.27) says that \( C_{\varepsilon} = 0 \), which in turn implies that

\[
g_{\varepsilon} = C' z F,
\]

\[
g_{\varepsilon\varepsilon} = C_{\varepsilon\varepsilon} + 2C_{\varepsilon} C' z F + C'' (z F)^2 - C' F' C_{\varepsilon\varepsilon}.
\]

- Compute \( C_{\varepsilon\varepsilon} \)

* Second-order terms in (13.7.28), we find that at \( \varepsilon = 0 \),

\[
\frac{d^3}{d\varepsilon^3} u_{00} C_{\varepsilon}^2 + \frac{d^3}{d\varepsilon^3} u_{00} C_{\varepsilon\varepsilon} = \beta \mathbb{E} \left\{ \frac{d^3}{d\varepsilon^3} g_{\varepsilon}^2 F' + 2u'' g_{\varepsilon} F' z - 2u'' g_{\varepsilon} F' z + u'' g_{\varepsilon} F' - 2u' z F'' C_{\varepsilon} + u' F'' C_{\varepsilon}^2 - u' F'' C_{\varepsilon\varepsilon} \right\}
\]

* Using the normalization \( \mathbb{E}\{z^2\} = 1 \), we find that

\[
\frac{d^2}{d\varepsilon^2} u_{00} C_{\varepsilon\varepsilon} = \beta \left[ \frac{d^3}{d\varepsilon^3} C' C' F^2 F' + 2u'' C' F F' + u'' (C_{\varepsilon\varepsilon} + C'' F^2 - C' F' C_{\varepsilon\varepsilon}) F' - u' F'' C_{\varepsilon\varepsilon} \right]
\]

* Solving for \( C_{\varepsilon\varepsilon} \) yields

\[
C_{\varepsilon\varepsilon} = \frac{\frac{d^2}{d\varepsilon^2} C' C' F^2 + 2u'' C' F + u'' C'' F^2}{\frac{d^2}{d\varepsilon^2} C' F' + \beta u' F''}
\]

- This exercise demonstrates that perturbation methods can also be applied to the discrete-time stochastic growth model.
Bifurcation Methods

- Suppose $H(h(\varepsilon), \varepsilon) = 0$ but $H(x, 0) = 0$ for all $x$.
  
  - IFT says
    
    $$h'(0) = -\frac{H_\varepsilon(x_0, 0)}{H_x(x_0, 0)}$$
  
  - $H(x, 0) = 0$ implies $H_x(x_0, 0) = 0$, and $h'(0)$ has the form $0/0$ at $x = x_0$.
  
  - l’Hospital’s rule implies, if which is well-defined if $H_{\varepsilon x}(x_0, 0) \neq 0$,
    
    $$h'(0) = -\frac{H_{\varepsilon\varepsilon}(x_0, 0)}{H_{\varepsilon x}(x_0, 0)}.$$
Example: Portfolio Choices for Small Risks

• Simple asset demand model:
  
  – safe asset yields $R$ per dollar invested and risky asset yields $Z$ per dollar invested
  
  – If final value is $Y = W((1 - \omega)R + \omega Z)$, then portfolio problem is
    \[
    \max_{\omega} E\{u(Y)\}
    \]

• Small Risk Analysis
  
  – Parameterize cases
    \[
    Z = R + \varepsilon z + \varepsilon^2 \pi
    \] (1)
  
  – Compute $\omega(\varepsilon) = \omega(0) + \varepsilon \omega'(0) + \frac{\varepsilon^2}{2} \omega''(0)$ around the deterministic case of $\varepsilon = 0$.
  
  – Failure of IFT: at $\varepsilon = 0$, $Z = R$, and $\omega(\varepsilon)$ is indeterminate, but we know that $\omega(\varepsilon)$ is unique for $\varepsilon \neq 0$
• Bifurcation analysis

- The first-order condition for $\omega$

\[
0 = E\{u' (WR + \omega W (\varepsilon z + \varepsilon^2 \pi)) (z + \varepsilon \pi)\} \equiv G(\omega, \varepsilon) \tag{2}
\]

\[
0 = G(\omega, 0), \quad \forall \omega. \tag{3}
\]

- Solve for $\omega(\varepsilon) = \omega(0) + \varepsilon \omega'(0) + \frac{\varepsilon^2}{2} \omega''(0)$. Implicit differentiation implies

\[
0 = G_\omega \omega' + G_\varepsilon \tag{4}
\]

\[
G_\varepsilon = E\{u''(Y)W(\omega z + 2\omega \varepsilon \pi)W(z + \varepsilon \pi) + u'(Y)\pi\} \tag{5}
\]

\[
G_\omega = E\{u''(Y)(z + \varepsilon \pi)^2 \varepsilon\} \tag{6}
\]

- At $\varepsilon = 0$, $G(\omega, 0) = G_\omega(\omega, 0) = 0$ for all $\omega$.

- No point $(\omega, 0)$ for application of IFT to (3) to solve for $\omega'(0)$.
• We want $\omega_0 = \lim_{\varepsilon \to 0} \omega(\varepsilon)$.

  - Bifurcation theorem keys on $\omega_0$ satisfying

    \[
    0 = G'_\varepsilon(\omega_0, 0) \\
    = u''(RW)\omega_0 \sigma^2_z W + u'(RW) \pi
    \]

    which implies

    \[
    \omega_0 = -\frac{\pi}{\sigma^2_z \frac{u'(WR)}{W u''(WR)}}
    \]  

  - (8) is asymptotic portfolio rule

    * same as mean-variance rule
    * $\omega_0$ is product of risk tolerance and the risk premium per unit variance.
    * $\omega_0$ is the limiting portfolio share as the variance vanishes.
    * $\omega_0$ is not first-order approximation.
To calculate $\omega'(0)$:

- Differentiate (2.4) with respect to $\varepsilon$

$$0 = G_{\omega\omega}\omega' + 2G_{\omega\varepsilon}\omega' + G_{\omega}\omega'' + G_{\varepsilon\varepsilon}$$  \hspace{1cm} (9)

where (without loss of generality, we assume $W = 1$)

$$G_{\varepsilon\varepsilon} = E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)^2(z + \varepsilon\pi) + u''(Y)2\omega\pi(z + \varepsilon\pi)$$
$$+ 2u''(Y)(\omega z + 2\omega\varepsilon\pi)\pi\}$$

$$G_{\omega\omega} = E\{u'''(Y)(z + \varepsilon\pi)^3\varepsilon\}$$

$$G_{\omega\varepsilon} = E\{u'''(Y)(\omega z + 2\omega\varepsilon\pi)(z + \varepsilon\pi)^2\varepsilon + u''(Y)(z + \varepsilon\pi)2\pi\varepsilon$$
$$+ u''(Y)(z + \varepsilon\pi)^2\}$$

- At $\varepsilon = 0$,

$$G_{\varepsilon\varepsilon} = u'''(R)\omega_0^2E\{z^3\} \quad G_{\omega\omega} = 0$$
$$G_{\omega\varepsilon} = u''(R)E\{z^2\} \neq 0 \quad G_{\varepsilon\varepsilon} \neq 0$$

- Therefore,

$$\omega' = -\frac{1}{2} \frac{u'''(R)E\{z^3\}}{u''(R)E\{z^2\}} \omega_0^2.$$  \hspace{1cm} (10)

- Equation (10) is a simple formula.

  * $\omega'(0)$ proportional to $u'''/u''$
  * $\omega'(0)$ proportional to ratio of skewness to variance.
  * If $u$ is quadratic or $z$ is symmetric, $\omega$ does not change to a first order.

- We could continue this and compute more derivatives of $\omega(\varepsilon)$ as long as $u$ is sufficiently differentiable.
Other applications - see Judd and Guu (ET, 2001)

- Equilibrium: add other agents, make $\pi$ endogenous
- Add assets
- Produce a mean-variance-skewness-kurtosis-etc. theory of asset markets
- More intuitive approach to market incompleteness then counting states and assets