Estimation of Static and Dynamic Models of Strategic Interactions

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This talk is based on the following papers


Motivation

1. Empirical analysis of games in econometrics and industrial organization.
2. Discrete choice model with other agent’s actions entering as a right hand side variable.
3. Often most straight forward to estimate a game in two steps.
4. In a first step, the economist estimates the reduced forms implied by the model.
5. In the second step, recover structural utility parameters that rationalize the observed reduced forms.
Literature

- Early examples, Vuong and Bjorn (1984) and Bresnahan and Reiss (1990, 1991).
- Single agent dynamic models: Rust (1987), Hotz and Miller (1993), Magnac and Thesmar (2003), Heckman and Navarro (2005), among many others.
Static Model

- Players, \( i = 1, \ldots, n \) and actions \( a_i \in \{0, 1, \ldots, K\} \) out of a finite set.
- Let \( A = \{0, 1, \ldots, K\}^n \) and \( a = (a_1, \ldots, a_n) \).
- Abstract from mixed strategies- unique best response.
- Let \( s_i \in S_i \) denote state for player \( i \).
- \( S = \prod_i S_i \) and \( s = (s_1, \ldots, s_n) \in S \).
- Assume that \( s \) is common knowledge and observed by econometrician.
- For each agent, \( K + 1 \) state variables \( \epsilon_i(a_i) \) which are private information to each agent.
• Density $f(\epsilon_i)$, i.i.d..
• Period utility for $i$: $u_i(a, s, \epsilon_i; \theta) = \Pi_i(a_i, a_{-i}, s; \theta) + \epsilon_i(a_i)$
• Utility similar to standard discrete choice model (e.g. multinomial logit).
• Parametric models: $\Pi_i(a_i, a_{-i}, s; \theta)$.
• Often a linear index in the literature.
• Unlike a standard discrete choice model, $a_{-i}$ enters utility.
• Generalizes a standard discrete choice model where agents act in isolation.
• Player $i$’s decision rule is a function $a_i = \delta_i(s, \epsilon_i)$.
• Note that $\epsilon_{-i}$ does not enter since this is private information of other players.
• Define conditional choice probability $\sigma_i(a_i|s)$ as:

$$\sigma_i(a_i = k|s) = \int 1\{\delta_i(s, \epsilon_i) = k\} f(\epsilon_i) d\epsilon_i.$$ 

• $\sigma_i(a_i = k|s)$ is the probability that $i$ chooses action $k$ conditional on the state variables $s$ that are public information.

• Define the “choice specific expected payoff” as

$$\Pi_i(a_i, s; \theta) = \sum_{a_{-i}} \Pi_i(a_i, a_{-i}, s; \theta)\sigma_{-i}(a_{-i}|s).$$

• Expected utility from $a_i$, not including preference shock.

• The optimal action for player $i$ satisfies:

$$\sigma_i(a_i|s) = Prob \left\{ \epsilon_i|\Pi_i(a_i, s; \theta) + \epsilon_i(a_i) > \Pi_i(a_j, s; \theta) + \epsilon_i(a_j) \text{ for } j \neq i \right\}.$$
• $\Pi_i(a_i, a_{-i}, s; \theta)$ is often a linear function, e.g.:

$$
\Pi_i(a_i, a_{-i}, s) = \left\{ \begin{array}{ll}
    s' \cdot \beta + \delta \sum_{j \neq i} \mathbf{1}\{a_j = 1\} & \text{if } a_i = 1 \\
    0 & \text{if } a_i = 0
\end{array} \right.
$$

• Mean utility from not entering normalized to zero.
• The term $\delta$ measures the influence of $j$’s choice on $i$’s entry decision.
• If profits decrease from having another firm enter the market then $\delta < 0$.
• The parameters $\beta$ measure the impact of the state variables on profits.
• The random error terms $\varepsilon_i(a_i)$ capture shocks to the profitability of entry.
• Choice specific expected payoff satisfies

\[ \Pi_i(a_i = 1, s; \theta) = s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s). \]

• Suppose that the error terms are distributed extreme value.

\[ \sigma_i(a_i = 1|s) = \frac{\exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s))}{1 + \exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s))} \]

• Full information maximum likelihood:
  - For each \( \beta \), solve for \( \sigma_j(a_j = 1|s; \beta) \) for all \( j = 1, \ldots, n \).
  - Maximize likelihood function:

\[
L(\beta, \delta) = \prod_{t=1}^{T} \prod_{i=1}^{n} \left( \frac{\exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s; \beta))}{1 + \exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s; \beta))} \right)^{1\{a_i, t=1\}} \left( 1 - \frac{\exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s; \beta))}{1 + \exp(s' \cdot \beta + \delta \sum_{j \neq i} \sigma_j(a_j = 1|s; \beta))} \right)^{1\{a_i, t=0\}}
\]
• Two step approach: replace $\sigma_j(a_j = 1|s; \beta)$ by $\hat{\sigma}_j(a_j = 1|s)$
• Pseudo likelihood easy to maximize: logit
• Both the first stage estimates $\hat{\sigma}_i(a_i = 1|s)$ and the term $s' \cdot \beta$
  depend on the vector of state variables $s$.
• Colinearity and identification: Need a covariate that enters the first stage, but not the second stage.
• Rest of paper: generalization
• Nonparametric Identification and estimation.
A1 Assume that the error terms $\epsilon_i(a_i)$ are distributed i.i.d. across actions $a_i$ and agents $i$, and come from a known parametric family.

- Not possible to allow nonparametric mean utility and error terms at once, even in simple single agent problems (e.g. a probit).
- In Bajari, Hong and Ryan (2005)- single agent model is not identified without an independence assumption.
- Well known that $\Pi_i(0, s)$ not identified: $\sigma_i(a_i|s)$ functions of $\Pi_i(a_i, s) - \Pi_i(0, s)$.
- Suppose $\epsilon_i(a_i)$ is extreme value,

$$
\sigma_i(a_i|s) = \frac{\exp(\Pi_i(a_i, s) - \Pi_i(0, s))}{\sum_{k=0}^{K} \exp(\Pi_i(k, s) - \Pi_i(0, s))}
$$
• Hotz and Miller (1993) inversion, for any $k$, $k'$:

$$\log(\sigma_i(k|s)) - \log(\sigma_i(k'|s)) = \Pi_i(k, s) - \Pi_i(k', s).$$

• More generally let $\Gamma : \{0, ..., K\} \times S \rightarrow [0, 1]$

$$(\sigma_i(0|s), ..., \sigma_i(K|s)) = \Gamma_i (\Pi_i(1, s) - \Pi_i(0, s), ..., \Pi_i(K, s) - \Pi_i(0, s))$$

• And the inverse $\Gamma^{-1}$:

$$(\Pi_i(1, s) - \Pi_i(0, s), ..., \Pi_i(K, s) - \Pi_i(0, s)) = \Gamma_i^{-1} (\sigma_i(0|s), ..., \sigma_i(K|s))$$

• Invert equilibrium choice probabilities to nonparametrically recover $\Pi_i(1, s) - \Pi_i(0, s), ..., \Pi_i(K, s) - \Pi_i(0, s)$.

• Can only learn choice specific value functions up to a first difference. Need normalization

$A2$ For all $i$ and all $a_{-i}$ and $s$, $\Pi_i(a_i = 0, a_{-i}, s) = 0$. 
• Similar to the “outside good” assumption in a single agent model.

• Entry: the utility from not entering is normalized to zero.

• \( \Pi_i(a_i, s) \) is known by our inversion and probabilities \( \sigma_i \) can be observed by econometrician.

• Next step: how to recover \( \Pi_i(a_i, a_{-i}, s) \) from \( \Pi_i(a_i, s) \).

• Requires inversion of the following system:

\[
\Pi_i(a_i, s) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s) \Pi_i(a_i, a_{-i}, s), \forall i = 1, \ldots, n, a_i = 1, \ldots, K.
\]

• Hold \( s \) fixed, \( n \times K \times (K + 1)^{n-1} \) unknowns utilities of all agents.

• Only \( n \times (K) \) known expected utilities.
• Obvious solution: impose exclusion restrictions.
• Partition $s = (s_i, s_{-i})$, and suppose
  $\Pi_i(a_i, a_{-i}, s) = \Pi_i(a_i, a_{-i}, s_i)$ depends only on the subvector $s_i$.

  $$\Pi_i(a_i, s_{-i}, s_i) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s_{-i}, s_i)\Pi_i(a_i, a_{-i}, s_i).$$

• Identification: Given each $s_i$, the second moment matrix of the “regressors” $\sigma_{-i}(a_{-i}|s_{-i}, s_i)$,

  $$E\sigma_{-i}(a_{-i}|s_{-i}, s_i)\sigma_{-i}(a_{-i}|s_{-i}, s_i)'$$

  is nonsingular.
• Needs at least $(K + 1)^{n-1}$ points in the support of the conditional distribution of $s_{-i}$ given $s_i$.
• Nonparametric Estimation a natural consequence of identification.
Introducing Dynamics

- Infinite Horizon, Stationary, Markov
- Conditional independence:
  - $\epsilon$ distributed i.i.d. over time.
  - State variables evolve according to $g(s'|s, a_i, a_{-i})$.
- Now players maximize expected discounted utility using discount factor $\beta$.

\[
W_i(s, \epsilon_i; \sigma) = \max_{a_i \in A_i} \left\{ \Pi_i(a_i, s, \epsilon_i) + \epsilon_i(a_i) \right. \\
+ \beta \int \sum_{a_{-i}} W_i(s', \epsilon'_i; \sigma) g(s'|s, a_i, a_{-i}) \sigma_{-i}(a_{-i}|s) f(\epsilon'_i) d\epsilon'_i \} 
\]

- Definition: A Markov Perfect Equilibrium is a collection of $\delta_i(s, \epsilon_i)$, $i = 1, \ldots, n$ such that for all $i$, all $s$ and all $\epsilon_i$, $\delta_i(s, \epsilon_i)$ maximizes $W_i(s, \epsilon_i; \sigma_i, \sigma_{-i})$. 
Players choose \( a_i \) to maximize \( V_i(a_i, s) + \epsilon_i(a_i) \).

Choice specific value function

\[
V_i(a_i, s) = \Pi_i(a_i, s) + \beta E\left[ V_i(s') | s, a_i \right].
\]

Ex ante value function

\[
V_i(s) = E_{\epsilon_i} \max_{a_i} [V_i(a_i, s) + \epsilon_i(a_i)]
\]

\[
= G(V_i(a_i, s), \forall a_i = 0, \ldots, K)
\]

\[
= G(V_i(a_i, s) - V_i(0, s), \forall a_i = 1, \ldots, K) + V_i(0, s)
\]

Mcfadden’s “social surplus function”.

When the error terms are extremely value distributed

\[
V_i(s) = \log \sum_{k=0}^{K} \exp(V_i(k, s))
\]

\[
= \log \sum_{k=0}^{K} \exp(V_i(k, s) - V_i(0, s)) + V_i(0, s). \]
• Relationship between \( \Pi_i(a_i, s) \) and \( V_i(a_i, s) \):

\[
V_i(a_i, s) = \Pi_i(a_i, s) + \beta E \left[ G(V_i(a_i, s'), \forall a_i = 0, \ldots, K) \mid s, a_i \right]
\]

\[
= \Pi_i(a_i, s) + \beta E \left[ G(V_i(k, s') - V_i(0, s'), \forall k = 1, \ldots, K) \mid s, a_i \right]
\]

\[
+ \beta E \left[ V_i(0, s') \mid s, a_i \right]
\]

• With extreme value distributed error terms

\[
V_i(a_i, s) = \Pi_i(a_i, s) + \beta E \left[ \log \sum_{k=0}^{K} \exp \left( V_i(k, s') - V_i(0, s') \right) \right] \mid s, a_i
\]

\[
+ \beta E \left[ V_i(0, s') \mid s, a_i \right]
\]

• Hotz and Miller (1993): choice probabilities \( \sigma_i(a_i|s) \) has a one to one relation to the choice specific value functions:

\[
(\sigma_i(0|s), \ldots, \sigma_i(K|s)) = \Gamma (V_i(1, s) - V_i(0, s), \ldots, V_i(K, s) - V_i(0, s))
\]

• Inverse mapping:

\[
(V_i(1, s) - V_i(0, s), \ldots, V_i(K, s) - V_i(0, s)) = \Omega_i (\sigma_i(0|s), \ldots, \sigma_i(K|s))
\]
Example: i.i.d extreme value $f(\epsilon_i)$:

$$
\sigma_i(a_i|s) = \frac{\exp(V_i(a_i, s) - V_i(0, s))}{\sum_{k=0}^{K} \exp(V_i(k, s) - V_i(0, s))}
$$

Inverse mapping:

$$
\log(\sigma_i(k|s)) - \log(\sigma_i(0|s)) = V_i(k, s) - V_i(0, s)
$$

If we know $V_i(0, s)$, $V_i(a_i, s)$ and $\Pi_i(a_i, s)$ is one to one.

Identify $V_i(0, s)$ first. Set $a_i = 0$:

$$
V_i(0, s) = \Pi_i(0, s) + \beta E \left[ \log \sum_{k=0}^{K} \exp(V_i(k, s') - V_i(0, s')) | s, 0 \right] + \beta E \left[ V_i(0, s') | s, 0 \right]
$$

This is a single contraction mapping unique fixed point iteration.

Add $V_i(0, s)$ to $V_i(k, s) - V_i(0, s)$ to identify all $V_i(k, s)$. 

• Then all $\Pi_i(k, s)$ calculated from $V_i(k, s)$.
• Why normalize $\Pi_i(0, s) = 0$?
• Why not $V_i(0, s) = 0$?
• If a firm stays out of the market in period $t$, current profit 0, but option value of future entry might depend on market size, number of other firms, etc.
• These state variables might evolve stochastically.
• Are there cases where $\Pi_i(0, s) \neq 0$?
• Is this an innocuous normalization?
• Rest of the identification arguments: identical to the static model.
Nonparametric estimation of static model:

- Estimation of Choice Probabilities.
- There are \( t = 1, \ldots, T \) repetitions of the game with actions and states \((a_{i,t}, s_{i,t}), i = 1, \ldots, n\).
- \( \hat{\sigma}_i(k|s) \) estimated using sieve series expansions (see Newey (1990) and Ai and Chen (2003)).
- Alternatively, kernel smoothing or local polynomial regressions.
- Let \( \{q_l(s), l = 1, 2, \ldots\} \) denote a sequence of known basis functions. Could use spline, Fourier Series or orthogonal polynomials.
- Denote the \( 1 \times \kappa(T) \) vector of basis functions as
  \[
  q^{\kappa(T)}(s) = (q_1(s), \ldots, q_{\kappa(T)}(s)),
  \]
  and
  \[
  Q_T = (q^{\kappa(T)}(s_1), \ldots, q^{\kappa(T)}(s_T)).
  \]
• Sieve Linear probability (or logit, probit etc)

\[ \hat{\sigma}_i(k|s) = \sum_{t=1}^{T} 1(a_{it} = k) q^{\kappa(T)}(s_t)(Q_T'Q_T)^{-1} q^{\kappa(T)}(s). \]

• Typically \( \hat{\sigma}_i(k|s) \) will converge to the true \( \sigma_i(k|s) \) at a nonparametric rate which is slower than \( T^{1/2} \).

• Second Step: Inversion

• Empirical analogue of the Hotz-Miller inversion between choice probabilities and choice specific value functions.

• Specific logit case,

\[ \hat{\Pi}_i(k, s_t) - \hat{\Pi}_i(0, s_t) = \log(\hat{\sigma}_i(k|s_t)) - \log(\hat{\sigma}_i(0|s_t)) \]

• Third Step: Recovering The Structural Parameters

• Form an estimate of \( \Pi(a_i, a_{-i}, s_i) \)
• Run a linear local weighted least squares regression, for each $i$:

$$
\sum_{t=1}^{T} \left( \frac{\hat{\Pi}_i(a_i, s_t) - \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{-it}, s_i) \Pi_i(a_i, a_{-i}, s_i)}{w(t, s_i)} \right)^2
$$

• $\hat{\Pi}_i(a_i, s_t)$ and $\hat{\sigma}_{-i}(a_{-i}|s_{-it}, s_{it})$ are taken as given from previous steps.

• The nonparametric weights $w(t, s_i)$ can take a variety of forms (e.g. kernel weights).

• Nonparametric approach is robust against misspecification but may suffer from a severe curse of dimensionality.

• Semiparametric alternative might be more practical for most applications.

$$
\Pi_i(a_i, a_{-i}, s_i) = \Phi_i(a_i, a_{-i}, s)' \theta
$$
• The optimal decision rule will take the form:

\[
\sigma_i(a_i|s; \theta) = \frac{\exp(\Phi_i(a_i, s)' \theta)}{\sum_{k=0}^{K} \exp(\Phi_i(k, s)' \theta)}
\]

\[
\Phi_i(k, s) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}|s_{-i}, s_i) \Phi_i(a_i, a_{-i}, s)
\]

• Nonparametric Estimate of the expected basis functions:

\[
\hat{\Phi}_i(k, s) = \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{-i}, s_i) \Phi_i(a_i, a_{-i}, s)
\]

• Pseudo MLE: use \(\hat{\Phi}_i(k, s)\) in place of \(\Phi_i(k, s)\).

• Or replace local weighted least square with global linear (weighted least square):

\[
\sum_{t=1}^{T} \left( \frac{\hat{\Pi}_i(a_i, s_t) - \sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{-it}, s_i) \Phi_i(a_i, a_{-i}, s)' \beta.}{\sum_{a_{-i}} \hat{\sigma}_{-i}(a_{-i}|s_{-it}, s_i) \Phi_i(a_i, a_{-i}, s)} \right)^2 w(t, s_t)
\]
• Extension to Dynamic models.
• Hotz-Miller inversion recovers \( V_i(k, s) - V_i(0, s) \) instead of \( \Pi_i(k, s) - \Pi_i(0, s) \).
• Nonparametrically compute \( V_i(0, s) \) using
  \[
  \hat{V}_i(0, s) = \beta \hat{E} \left[ \log \sum_{k=0}^{K} \exp \left( \hat{V}_i(k, s') - \hat{V}_i(0, s') \right) | s, 0 \right] 
  + \beta \hat{E} \left[ \hat{V}_i(0, s') | s, 0 \right]
  \]
• Obtain and \( \hat{V}_i(k, s) \) and forward compute \( \hat{\Pi}_i(k, s) \).
• The rest is identical to the static model.
• Apply the results of Newey (1994)-derive appropriate “influence functions”.

• This guarantees that the model converges \( \theta \) at a \( T^{1/2} \) rate and has normal asymptotics.

• The asymptotic distribution is invariant to the choice of method used to estimate the first stage.

• With proper weighting function (need to estimate nonparametrically), can achieve the same efficiency as full information maximum likelihood.

• Above statements hold for both static and dynamic models.
Extensions

- Binary choice linearity probability model: can be estimated using 2SLS in Stata.
- Developed an algorithm to find all equilibrium for the static model.
- The “all solutions homotopy”.
- Verify the regularity conditions required to demonstrate that all solutions are found.
- Useful since games do not generally predict unique solutions.
- Multiple equilibria important when simulating the model.
- Fixed effect type unobserved heterogeneity- Chamberlain’s conditional logit or maximum score ideas.
- Need fixed effects to be a smooth function of $s$. 
Computing Multiple Equilibria

• Given known distribution for the error term $F(\epsilon_i)$ and the mean utility functions $\Pi_i(a_i, a_{-i}, \theta)$, conditional choice probabilities defined by fixed point mappings:

$$\sigma_i(a_i|s) = \Gamma_i \left( \sum_{a_{-i}} \sigma_{a_{-i}} (a_{-i}|s) \left[ \Pi_i(k, a_{-i}, s; \theta) - \Pi_i(0, a_{-i}, s; \theta) \right], k = 1, \ldots, K \right)$$

• Given linear mean utility, fixed point mappings:

$$\sigma_i(a_i|s) = \Gamma_i \left( \sum_{a_{-i}} \sigma_{a_{-i}} (a_{-i}|s) \Phi_i(a_i, a_{-i}, s)' \theta, a_i = 1, \ldots, K \right), \quad i = 1, \ldots, n.$$  

• $K \times n$ equations and $K \times n$ unknown variables

$$\sigma_i(a_i|s), \forall a_i = 1, \ldots, K, \quad i = 1, \ldots, n.$$
Homotopy Method

• Find all solutions to the fixed point system, for $\sigma = \sigma (s)$:
  \[ \sigma - \Gamma (\sigma) = 0, \]

• Homotopy: linear mapping of the form
  \[ H(\sigma, \tau) = \tau G(\sigma) + (1 - \tau)(\sigma - \Gamma (\sigma)), \quad \tau \in [0, 1], \]

• $H(\sigma, \tau)$ and $G(\sigma)$: vectors of functions with $n \times K$ component functions
  \[ H_{i,a_i}(\sigma, \tau) \text{ and } G_{i,a_i}(\sigma) \text{ for } i = 1, \ldots, n \text{ and } a_i = 1, \ldots, K. \]

• $\tau = 0$: $H(\sigma, 0) = \Gamma (\sigma)$. $\tau = 1$: $H(\sigma, 0) = G(\sigma)$.

• At each $\tau$, denote the solution along a path by $\sigma (\tau)$:
  \[ H(\sigma (\tau), \tau) = 0. \]

• Differentiating this homotopy with respect to $\tau$:
  \[ \frac{d}{d\tau} H(\sigma (\tau), \tau) = \frac{\partial H}{\partial \tau} + \frac{\partial H}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \tau}. \]

• Algorithms for numerically tracing this differential equation system.
Multiple Equilibria in Discrete Games

- All solution homotopy from polynomial system of equations.
- Choice of the initial system

$$G_{i,a_i}(\sigma) = \sigma_i(a_i)^{q_{i,a_i}} - 1 = 0 \quad \text{for} \quad i = 1, \ldots, n \quad \text{and} \quad a_i = 1, \ldots, K,$$

- The resulting homotopy mapping

$$H_{i,a_i}(\sigma, \tau) = \tau\{\sigma_i(a_i)^{q_{i,a_i}} - 1\} + (1-\tau)\left(\sigma_i(a_i) - \Gamma_{i,a_i}(\sigma)\right), \quad \tau \in [0, 1].$$

**Theorem**

For given $\tau$ one can pick the power $q_{i,a_i}$ of the initial function such that the homotopy system is regular and path finite given some sequence of converging polyhedra $\varnothing_\epsilon$, $\epsilon \to 0$. 
Theorem
Define the sets $H^{-1} = \{(\sigma_r, \sigma_i, \tau) \mid H(\sigma_r, \sigma_i, \tau) = 0\}$ and

\[ H^{-1}(\tau) = \{(\sigma_r, \sigma_i) \mid H(\sigma, \tau) = 0\} \quad \text{for} \quad \sigma_r \in \mathbb{R}^{nK}, \quad \text{and} \quad \sigma_i \in \mathbb{R}^{nK}. \]

Note that $H$ is a homotopy of dimension $\mathbb{R}^{2nK}$ that include both real and imaginary parts separately. Also define, for any small $\epsilon$, $\& \epsilon = \bigcup_{i,a_i} \{ |\sigma_{r,i,a_i}| \leq \epsilon \}$ to be the area around the imaginary axis. Then: 1) The set $H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \& \epsilon \times [0, 1]\}$ consists of closed disjoint paths.

2) For any $\tau \in (0, 1]$ there exists a bounded set such that $H^{-1}(\tau) \cap \mathbb{R}^{2nK} \setminus \& \epsilon$ is in that set.

3) For $(\sigma_r, \sigma_i, \tau) \in H^{-1} \cap \{\mathbb{R}^{2nK} \setminus \& \epsilon \times [0, 1]\}$ the homotopy system allows parametrization $H(\sigma_r(s), \sigma_i(s), \tau(s)) = 0$. Moreover, $\tau(s)$ is a monotone function.
Monte Carlo analysis

- Entry game with a small number of players
- Multiple equilibria computation, not about identification.
- Payoff to player $i$ a linear function of the indicator of the rival’s entry ($a_i = 1$), market covariates and random term:

$$U_i(1, a_{-i}) = \theta_1 - \theta_2 \left( \sum_{j \neq i} 1(a_j = 1) \right) + \theta_3 x_1 + \theta_4 x_2 + \epsilon_i(a),$$

$$i = 1, \ldots, n.$$

- Symmetric model, ex-ante probability of entry:

$$P_i = \frac{e^{\theta_1-\theta_2(\sum_{j \neq i} P_j)+\theta_3 x_1+\theta_4 x_2}}{1 + e^{\theta_1-\theta_2(\sum_{j \neq i} P_j)+\theta_3 x_1+\theta_4 x_2}}$$
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<th>Distribution</th>
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Table: Results of Monte-Carlo Simulations.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$n = 3$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td># of equilibria</td>
<td>1.592</td>
<td>1.175</td>
<td>7</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.366</td>
<td>0.362</td>
<td>0.998</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.360</td>
<td>0.367</td>
<td>0.995</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.363</td>
<td>0.348</td>
<td>0.993</td>
<td>0.003</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$n = 4$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td># of equilibria</td>
<td>1.292</td>
<td>0.777</td>
<td>5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.278</td>
<td>0.328</td>
<td>0.981</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.246</td>
<td>0.320</td>
<td>0.981</td>
<td>0.003</td>
<td></td>
</tr>
<tr>
<td>$P_3$</td>
<td>0.276</td>
<td>0.338</td>
<td>0.999</td>
<td>0.001</td>
<td></td>
</tr>
<tr>
<td>$P_4$</td>
<td>0.280</td>
<td>0.338</td>
<td>0.987</td>
<td>0.002</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$n = 5$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td># of equilibria</td>
<td>1.106</td>
<td>0.505</td>
<td>5</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$P_1$</td>
<td>0.104</td>
<td>0.201</td>
<td>0.964</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$P_2$</td>
<td>0.138</td>
<td>0.252</td>
<td>0.975</td>
<td>0</td>
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<tr>
<td>$P_3$</td>
<td>0.315</td>
<td>0.338</td>
<td>0.992</td>
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</tr>
<tr>
<td>$P_4$</td>
<td>0.356</td>
<td>0.385</td>
<td>0.983</td>
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<tr>
<td>$P_5$</td>
<td>0.319</td>
<td>0.344</td>
<td>0.982</td>
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</table>
Table: Frequencies for the numbers of equilibria.

<table>
<thead>
<tr>
<th># of equilibria</th>
<th>Number of cases</th>
<th>Frequency (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>192</td>
<td>47.93</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>132</td>
<td>33.06</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>64</td>
<td>16.12</td>
</tr>
<tr>
<td>$n = 7$</td>
<td>12</td>
<td>2.89</td>
</tr>
<tr>
<td>Total</td>
<td>400</td>
<td>100</td>
</tr>
<tr>
<td>$n = 4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>287</td>
<td>71.84</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>93</td>
<td>23.30</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>20</td>
<td>4.85</td>
</tr>
<tr>
<td>Total</td>
<td>400</td>
<td>100</td>
</tr>
<tr>
<td>$n = 5$</td>
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<td></td>
</tr>
<tr>
<td>$n = 1$</td>
<td>373</td>
<td>93.16</td>
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<tr>
<td>$n = 3$</td>
<td>25</td>
<td>6.21</td>
</tr>
<tr>
<td>$n = 5$</td>
<td>2</td>
<td>0.62</td>
</tr>
<tr>
<td>Total</td>
<td>400</td>
<td>100</td>
</tr>
</tbody>
</table>
Table: Tabulation of Probability of entry of the first player.

<table>
<thead>
<tr>
<th># of equilibria</th>
<th>n = 3</th>
<th></th>
<th></th>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std Dev</td>
<td>Max</td>
<td>Min</td>
<td></td>
</tr>
<tr>
<td>n = 1</td>
<td>.375</td>
<td>.386</td>
<td>.998</td>
<td>0</td>
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</tr>
<tr>
<td>n = 3</td>
<td>.337</td>
<td>.341</td>
<td>.978</td>
<td>.001</td>
<td></td>
</tr>
<tr>
<td>n = 5</td>
<td>.353</td>
<td>.322</td>
<td>.936</td>
<td>.006</td>
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</tr>
<tr>
<td>n = 7</td>
<td>.601</td>
<td>.367</td>
<td>.957</td>
<td>.050</td>
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</table>

<table>
<thead>
<tr>
<th># of equilibria</th>
<th>n = 4</th>
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<tbody>
<tr>
<td></td>
<td>Mean</td>
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<td>Max</td>
<td>Min</td>
<td></td>
</tr>
<tr>
<td>n = 1</td>
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<td>.300</td>
<td>.981</td>
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<tr>
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<td>.029</td>
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<td>.235</td>
<td>.551</td>
<td>.021</td>
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</table>

<table>
<thead>
<tr>
<th># of equilibria</th>
<th>n = 5</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Std Dev</td>
<td>Max</td>
<td>Min</td>
<td></td>
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<tr>
<td>n = 1</td>
<td>.116</td>
<td>.216</td>
<td>.964</td>
<td>.002</td>
<td></td>
</tr>
<tr>
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<td>.665</td>
<td>.001</td>
<td></td>
</tr>
<tr>
<td>n = 5</td>
<td>.007</td>
<td>.232</td>
<td>.436</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>