Dynamic Programming with Piecewise Linear Interpolation

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Piecewise Linear Interpolation
If Lagrange data \(\{(x_i, v_i) : i = 1, \ldots, m\}\) is given, then its piecewise linear interpolation is
\[
\hat{V}(x) = b_{j,0} + b_{j,1}x, \quad \text{if } x \in [x_j, x_{j+1}],
\]
where
\[
b_{j,1} = \frac{v_{j+1} - v_j}{x_{j+1} - x_j},
\]
\[
b_{j,0} = v_i - b_{j,1}x_i,
\]
for \(j = 1, \ldots, m - 1\).
In the maximization step of numerical DP algorithms, one could directly solve the maximization problem

\[ v_i = \max_a u(x_i, a) + \beta \hat{V}(y; b^+) \]

where

\[ y = g(x_i, a) \]

Problem: \( \hat{V}(x; b^{t+1}) \) is not differentiable, making it difficult to solve the optimization problem for \( a \).
Min-Function Approach

- The differentiability problem is solved as follows:

\[ v_i = \max_{a, w, y} u(x_i, a) + \beta \ w \]

s.t. \[ y = g(x_i, a) \]

\[ w \leq b_{j,0}^+ + b_{j,1}^+ y, \quad 1 \leq j < m \]

- optimization solvers can still solve the new model quickly

  - The objective function is smooth
  - inequality constraints are linear and sparse
  - we can apply fast Newton-type optimization algorithms to solve this problem if \( g \) is also smooth.
  - although this new model adds \( (m - 1) \) linear inequality constraints, few of them will be active at any iteration

- this way does not need to find the interval where \( y \) locates, while the spline approximation of value function must
Convex-Set Approach

- Both previous methods need to calculate coefficients; this is very complicated for multi-dimensional piecewise linear interpolation.
- Convex set approach never computes coefficients of approximation:

\[
\begin{align*}
    v_i &= \max_{\mu_j \geq 0, a, w, y} \mu_j x_j^{+} + u(x_i, a) + \beta w, \\
    \text{s.t.} \quad y &= g(x_i, a), \\
    y &= \sum_{j=1}^{m} \mu_j x_j^{+}, \\
    w &\leq \sum_{j=1}^{m} \mu_j v_j^{+}, \\
    \sum_{j=1}^{m} \mu_j &= 1
\end{align*}
\]