Dynamic Programming for Portfolio Problems

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Proportional Transaction Cost and CRRA Utility

- Separability of wealth $W$ and portfolio fractions $x$.
- If $u(W) = W^{1-\gamma}/(1 - \gamma)$, then $V_t(W_t, x_t) = W_t^{1-\gamma} \cdot g_t(x_t)$.
- If $u(W) = \log(W)$, then $V_t(W_t, x_t) = \log(W_t) + \psi_t(x_t)$.
- "No-trade" region: $\Omega_t = \{x_t : (\delta_t^+)^* = (\delta_t^-)^* = 0\}$, where $(\delta_t^+)^* \geq 0$ are fractions of wealth for buying stocks, and $(\delta_t^-)^* \geq 0$ are fractions of wealth for selling stocks.
Examples with two stocks

We first look at simple examples:

- Two stocks and one bond
- Investment begins at \( t = 0 \) and is liquidated at \( t = 6 \)
i.i.d. returns (uniform distribution on [0.87, 1.27])

No-trade region at stage $t=0$
No-trade region at stage $t=3$
No-trade region at stage $t=5$
Correlated returns (discrete)

- **Return:** \( R_1 = 0.85, 1.08, 1.25; \) \( R_2 = 0.88, 1.06, 1.2; \)

- **Joint Probability Matrix:**
  \[
  \begin{bmatrix}
  0.08 & 0.1 \\
  0.12 & 0.4 & 0.12 \\
  0.1 & 0.08 \\
  \end{bmatrix}
  \]

No-trade region at stage t=0
No-trade region at stage $t=3$
No-trade region at stage $t=5$
Stochastic mean return

- Log-normal return: \( N\left(\mu_i - \frac{\sigma_i^2}{2}, \sigma_i^2\right) \) with \( \mu_i = 0.06 \) or \( 0.08 \) and \( \sigma_i = 0.2 \), for \( i = 1, 2 \)

- Transition probability matrix of \( \mu \):

\[
\begin{bmatrix}
0.75 & 0.25 \\
0.25 & 0.75
\end{bmatrix}
\]

Stock 1 fraction

Stock 2 fraction

No-trade region at stage \( t=0 \)
No-trade region at stage $t=3$
No-trade region at stage $t=5$
Three stocks

- Problem: 3 stocks + 1 bond.
- Investment begins at $t = 0$ and is liquidated at $t = 6$
- Log-normal return: $\mathcal{N} \left( \mu - \sigma^2/2, \Lambda \Sigma \Lambda \right)$ with $\sigma = (0.16, 0.18, 0.2)^T$, 
  $\mu = (0.07, 0.08, 0.09)^T$, $\Lambda$ is the diagonal matrix of $\sigma$, and
  $\Sigma = \begin{bmatrix} 1 & 0.2 & 0.1 \\ 0.2 & 1 & 0.314 \\ 0.1 & 0.314 & 1 \end{bmatrix}$.
- degree-7 complete Chebyshev approximation
- product Gauss-Hermite quadrature rule with 9 nodes in each dimension
Problem: 3 stocks + 1 bond, 50 periods

No-trade regions:

\[ \Omega_{49} \approx [0.16, 0.33]^3, \Omega_{48} \approx [0.14, 0.24]^3, \ldots, \]
\[ \Omega_t \approx [0.19, 0.27]^3, \text{ for } t = 0, \ldots, 15, \]

Merton’s ratio: \[ x^* = (\Lambda \Sigma \Lambda)^{-1}(\mu - r)/\gamma = (0.25, 0.25, 0.25)^\top \]
$\Sigma_{12} = \Sigma_{13} = 0.2, \Sigma_{23} = 0.04$
More correlation: $\Sigma_{12} = \Sigma_{13} = 0.4, \Sigma_{23} = 0.16$
Portfolios with options

- Problem: 1 stock, 1 put option on the stock, and 1 bond
- Investment begins at $t = 0$ and is liquidated at $t = 6$ months
- Proportional transaction costs in stock and option trades
- Return of option is based on pricing of the option.
  - We use the binomial lattice method to price the option
  - Stock price will be one exogenous state variable
- Separability of wealth $W$ and portfolio fractions $x$ and stock price $S$
  - If $u(W) = W^{1-\gamma}/(1 - \gamma)$, then $V_t(W_t, S_t, x_t) = W_t^{1-\gamma} \cdot g_t(S_t, x_t)$.
  - If $u(W) = \log(W)$, then $V_t(W_t, S_t, x_t) = \log(W_t) + \psi_t(S_t, x_t)$. 
1 stock and 1 at-the-money put option at \( t = 0 \) (liquidate at \( t = 6 \) months)

- **Put option**: strike \( K \), expiration time \( T \), payoff \( \max(K - S_T, 0) \)
- **Stock price** \( S \), utility \( u(W) = -W^{-2}/2 \)
Dependence on transaction costs

1 at-the-money put option at $t = 0$

transaction cost ratios: $\tau_1 = 0.01, \tau_2 = 0.02$
1 at-the-money put option at $t = 0$

transaction cost ratios: $\tau_1 = 0.01$, $\tau_2 = 0.005$
1 at-the-money put option at $t = 0$ (liquidate at $t = 6$)

transaction cost ratios: $\tau_1 = 0.005$, $\tau_2 = 0.01$
1 at-the-money put option at $t = 0$

transaction cost ratios: $\tau_1 = 0.005, \tau_2 = 0.005$
Dependence on horizon

1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$)

expiration time: $T = 1$ month
1 at-the-money put option ($\tau_1 = 0.01$, $\tau_2 = 0.01$)
expiration time: $T = 2$ months
1 at-the-money put option \((\tau_1 = 0.01, \tau_2 = 0.01)\)
expiration time: \(T = 3\) months
1 at-the-money put option ($\tau_1 = 0.01, \tau_2 = 0.01$)
expiration time: $T = 4$ months
1 at-the-money put option ($\tau_1 = 0.01, \tau_2 = 0.01$)

expiration time: $T = 5$ months
Dependence on strike price \( (K = 0.8S_0) \)

1 put option at \( t = 0 \) (liquidate at \( t = 6, \tau_1 = \tau_2 = 0.01 \))
$K = 1.2S_0$

1 put option at $t = 0$ (liquidate at $t = 6$, $\tau_1 = \tau_2 = 0.01$)
Social value of options

Value functions with/without options at \( t = 0 \) (liquidate at \( t = 6 \))

Value Functions \( V_0(W,S,x,y) \) at \( W = 1, S = 1 \) and \( y = 0 \)

\( \tau_1 = 0.01 \) and \( \tau_2 \): transaction cost ratios of stock and option

\( (x, y) \): fractions of money in stock and option
Summary

- Developed a NDP method with shape-preserving approximation, stabler
- Developed a NDP method with Hermite interpolation, more accurate and more time-saving
- Developed a parallel NDP algorithm, running over hundreds of computers, almost linear speed-up
- Solved arbitrary-number-of-period and many-asset dynamic portfolio problems with transaction costs
- Solved arbitrary-number-of-period dynamic portfolio problems with options and transaction costs